Synthesis/Programming of Hopfield Associative Memory

Rama Murthy Garimella, Krishna Vamshi Reddy Lankala, Devaki Nimmagadda and Sree Divya Bondalapati

November 11, 2019

Synthesis/Programming of Hopfield Associative Memory

Garimella Rama Murthy[1], Lankala Krishna Vamshi Reddy[2], Devaki Nimmagadda[3] and Sree Divya Bondalapati[4]
Mahindra Ecole Centrale – Dept of Computer Science
Hyderabad, Telangana - India

Abstract: In this research paper, synthesis of Hopfield Associative memory with desired orthogonal stable states is discussed when the threshold vector is non-zero (and the dimension, N is even). Synthesis of one desired stable state in Hopfield Associative Memory (HAM) when N is odd is discussed. Based on relationship between Hopfield Associative Memory and associated graph-theoretic code, noise immunity issues are discussed.

1 Introduction

In an effort to model the biological neuronal cell, McCulloch and Pitts proposed a model of artificial neuron. It was able to separate patterns belonging to two classes using a linear decision boundary i.e. hyperplane. Thus, it was realized that the model enables recognition of patterns belonging to two classes. But the model does not have "learning ability" since the synaptic weights are all kept constant throughout the classification process. Rosenblatt proposed the idea of varying the synaptic weights using a "learning law" which converges when the patterns are linearly separable. In an effort to enable classification of "non-linearly" separable patterns, Werbos proposed the multi-layer perceptron which was trained using the "backpropagation algorithm".

Hopfield, in an effort to model biological memory, proposed the so called, Hopfield Associative Memory (HAM). Such a Hopfield Neural Network (HNN) is based on the Mc-Culloch-Pitts model of artificial neuron. In such an Artificial Neural Network (ANN), the state space constitutes the symmetric unit hypercube (i.e. the components of state vectors are {+1 or -1}). The associated convergence theorem ensures that in the serial mode of operation, the initial state converges to a stable state and in the fully parallel mode of operation, a cycle of length at most 2 is reached. In essence, the stable states are realized as the "memory states" of the associative memory. Hopfield was naturally interested in synthesizing a HAM, with certain "desired stable states" (that are preselected). This initial effort led to many efforts by the researchers to solve the SYNTHESIS/ PROGRAMMING problem associated with the HAM. This research paper is an effort in that direction. This research paper is organized as follows. In Section 2, related research work is reviewed. In Section 3, synthesis in even dimension is discussed. In Section 4, synthesis in odd dimension is discussed. In Section 5, noise immunity of HAM is explored. The research paper concludes in section 6.

2 Review of Related Literature

Hopfield Neural Network (HNN) is a network of nodes/neurons/units which are connected to each other. At any point of time, state of each unit is either +1 or -1. Links between the units are symmetric in any direction and carry synaptic weights. For example, consider two nodes A, B connected by a link carrying a weight(w). Here, the link corresponding from A to B(w_{ab}) is same as the link from B to A(w_{ba}) and $w_{ab} = w_{ba}$. We have a node network. W is the synaptic weight matrix. W is a symmetric matrix whose diagonal elements are all non-negative.

Hopfield proposed an outer-product rule to store certain orthogonal vectors lying on the unit hypercube as the “desired memories”. This approach was successfully utilized by researchers in synthesizing an associative memory. But, it was realized by researchers that the approach proposed by Hopfield results in exponentially many “SPURIOUS” stable states [BrR]. The author understood the logical basis of “desired stable state synthesis” and provided a very general synthesis approach in [RaM], when ‘N’ is even, and threshold vector, $\overline{T} \equiv 0$.

3 Synthesis of Hopfield Associative Memory in Even Dimension

In [RaM], it was assumed that when the synaptic weight matrix is synthesized using the orthogonal corners of the hypercube as stable/anti-stable states , the HNN is said to exist. In [BrR], it was not realised that orthogonal stable states exist only when the dimension is even.

3.1 Lemma 1: Consider two vectors X,Y. We compute X^TY, when X,Y are {+1,-1} vectors and $X \neq Y$. $X^TY = -(N-2K)$ where ‘N’ be the dimension of X, Y and ‘K’ be the number of places X, Y agree with each other.

Proof: Given ‘N’, the dimension of X, Y and ‘K’ be the number of places X and Y agree with each other, it implies that ‘N-K’ is the number of places they differ.

X^TY = {number of places X, Y agree} – {number of places X,Y differ}
= K – (N – K), for N > K
= – (N – 2K)

If ‘N’ is an even integer, then X^TY takes values from {–N, – (N – 2), – (N – 4), ….., 0, 2, 4, ….., (N – 2)} for values of K = {0,1, 2, ….., $(\frac{N}{2})$, $[\frac{N}{2}+1]$, $[\frac{N}{2}+2]$, ….., (N – 2)}. Similarly, if ‘N’ is an odd integer, then X^TY takes values from {–N, – (N – 2), – (N – 4), ….., 1, 3, …..,(N – 1)} for values of K = {0,1, 2, ….., $[\frac{N}{2}+1]$, $[\frac{N}{2}+2]$, ….., (N – 1)}.

Corollary: From the above lemma, X^TY is equal to zero when ‘K = $\frac{N}{2}$’. This specifies the condition for the two vectors to be orthogonal to each other.

Definition: Two vectors A, B are said to be orthogonal if their inner product is zero, i.e., <A, B>=0.

3.2 Spectral Representation of Symmetric Matrix

Given V_i , a {+1, –1} component eigenvector corresponding to eigenvalue λ_i, the synaptic weighted matrix $W = \sum_{i=1}^{N} \lambda_i V_i V_i^T$, where V_i is a column vector and V_i^T is a row matrix. Then,

$$\text{Sign}\{W V_i\} = \text{Sign}\{\lambda_i V_i\} = \text{Sign}\{V_i\} = V_i$$

3.3 Stable States

If V is an eigen vector corresponding to positive eigen value (also corner of hyper cube) then V is a stable state, when threshold vector $(\overline{T}) \equiv 0$.

3.4 Orthogonal Stable States: Hadamard Matrix

To synthesize associative memory in even dimension, the corners of the hypercube (any pair) should be mutually orthogonal {+1, -1} with each other, and these orthogonal corners are said to be the desired stable states. To attain this, Hadamard matrices are found to be useful. A Hadamard matrix is a matrix containing +1's and -1's and is denoted by H_n, where 'n x n' is the order of the matrix. A Hadamard matrix(H_n) of order 'n' satisfies the following equation '$H_n \times H^T_n = n I_n$'.

Sylvester construction enables synthesis of Hadamard matrices of order $n = 2^m$ for an integer 'm'. Thus, orthogonal stable states can be chosen as columns of Hadamard matrix. But dimension, of H_n must be a power of two.

3.5 Non-zero Threshold vector: Synthesis of W

In [RaM], when the dimension, N is even (i.e., W is an "N x N" matrix), HAM is synthesized using orthogonal corners of the unit hypercube. The synthesis assumed that the threshold vector, is a vector of all zeroes.

Suppose the threshold vector is a non-zero vector (i.e., at least some components are non-zero).

Case 1: Let W be a positive definite matrix i.e., all the eigenvalues are positive i.e., $\lambda_{max} > \lambda_{N-1} > \ldots > \lambda_{min} = \lambda_1$ we should choose $\lambda_{min} > |T_i|$ for all 'i', i.e., λ_{min} is larger than the maximum absolute value of components threshold vector $\overline{T}$. $\overline{U}_i$ be an eigenvector of W corresponding to eigenvalue "λ_i".

We readily have that, $\text{Sign}\{W\overline{U}_i - \overline{T}\} = \text{Sign}\{\lambda_i \overline{U}_i - \overline{T}\} = \overline{U}_i$ for all $1 \leq i \leq N$. Thus, even when $\overline{T} \neq 0$, synthesis of $\overline{W}$ with desired stable states can be done since there is freedom in the choice of eigenvalues.

Suppose some eigenvalues of $\overline{W}$ are negative i.e., Say $\{-\mu_1, -\mu_2, \ldots, -\mu_s\}$ are negative eigenvalues and $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_{N-s}\}$ are positive eigenvalues.

Let, $\mu_{max} = \text{Max } \{|\mu_1|, |\mu_2|, \ldots\ldots., |\mu_s|\}$ and $\lambda_{min} = \text{Min } \{\lambda_1, \lambda_2, \ldots\ldots, \lambda_{N-s}\}$. We now choose $\mu_{max} < |T_i|$ for all $1 \leq i \leq N$.

Let 'V_i' be an eigenvector corresponding to negative eigenvalues "$-\mu_i$". We readily have $\text{Sign}\{W\overline{V}_i - \overline{T}\} = \text{Sign}\{-\mu_i \overline{V}_i - \overline{T}\} = -\overline{V}_i$, for all $1 \leq i \leq N$

The synthesis procedure discussed needs orthogonal corners of hypercube. Hadamard matrix columns readily provide them. But Sylvester construction of Hadamard matrices requires than $N=2^n$ for an integer 'n'.

Note:

a) The Hamming distance between any pair of desire stable states (out of N of them) is $\frac{N}{2}$. We conjecture it to be maximum possible value. Thus synthesis is optimal.

b) In even dimension (i.e. N is even) if two desired stable states are eigenvectors, all of the eigenvectors must be stable states.
Due to lack of space proof of "optimality" is not discussed.

4 Synthesis of Hopfield Associative Memory in Odd Dimension

It was concluded in [RaM] that Hopfield type synthesis of HAM is NOT possible in odd dimension i.e., 'N' is odd (in fact, it was claimed that Hopfield Associative Memory doesn't exist in the odd dimension). This conclusion requires clarification.

Since, two orthogonal {+1, -1} vectors lying on unit hypercube don't exist, when N is odd, at most ONE corner of hypercube can be "DESIRED" stable state. It is an eigenvector corresponding to one of the positive eigenvalues. All other orthogonal eigenvectors of W will not lie on the unit hypercube.

Suppose the corner of hypercube which is also an eigenvector corresponding to positive eigenvalue happens to $\overline{e}$ i.e., all - ONES vector. In such case all other eigenvectors are constrained (as all the eigenvectors must form an orthonormal basis). For example, if $\overline{f}$ is any other eigenvector. We must have that Sum of positive components of $\overline{f}$ = sum of negative components of $\overline{f}$. Thus 'L^1 – norm ($\overline{f}$)' must be divisible by two (since the eigenvector form an orthonormal basis).

Suppose $\overline{W}$ is Laplacian i.e., '0' is an eigen value (of multiplicity at least one). Then $\overline{W}\,\overline{e} = \overline{0} \Rightarrow \text{Sign}(\overline{W}\,\overline{e}) = +\overline{e}$ or $-\overline{e}$ depending on whether we denote Sign(0) = +1 or –1. Hence $\{+\overline{e}, -\overline{e}\}$ are stable states.

Similarly, suppose $\overline{h}$ lies in the null space of W i.e., $\overline{W}\,\overline{h} = 0$. Then $\text{Sign}\{\overline{W}\,\overline{h}\} = +\overline{e}$ or $-\overline{e}$ i.e., $\overline{h}$ lies in the domain of attraction of stable states $+\overline{e}$ or $-\overline{e}$. In fact, all the corners of hypercube in Null space of $\overline{W}$ lie in the domain of attraction of the vectors $\{+\bar{e}$ or $-\bar{e}\}$ depending on the choice of notation.

4.1 One desired stable state of Hopfield Associative Memory

From the earlier discussion, there are no orthogonal {+1, –1} vectors (lying on the unit hypercube) when the dimension of the vector is odd. Thus, there can be at most on corner of the hypercube (suitably normalized) that can be an eigenvector (corresponding to a suitably chosen eigenvalue) in the spectral representation of symmetric matrix, W. Hence only one stable state can be chosen as the desired memory. As per the earlier classification, all the other stable states are "SPURIOUS". Such a synthesis of Hopfield Neural Network can be interpreted as "ONE DESIRED STABLE STATE ASSOCIATIVE MEMORY" i.e., such a memory can be effectively utilized to store exactly one desired pattern. In fact, using similar idea, we can design associative memories which store K, $1 \leq K \leq N$ desired stable states when the dimension, N is EVEN.

Note:

a) Using the results in [BrB], it is possible to arrive at ALL the set of STABLE STATES associated with a synaptic weight matrix, W (as well as the associated graph). Specifically, the set of codewords of the associated graph - theoretic code (based on the cuts in the graph) determine the stable states. So far, in the odd dimension, programming/synthesis of a Hopfield Network (if possible "optimal" in the sense of maximizing minimum distance between a collection of chosen stable states) with CHOSEN STABLE STATES has not been investigated thoroughly. We are currently investigating such problem.

b) We are interested in the distance (specifically minimum distance) between stable states corresponding to codewords of the graph - theoretic code (not so much the minimum distance of the graph theoretic code based on W & the graph).

5 Noise Immunity of Hopfield Associative Memory

Bruck [BrB] showed the relationship between HAM and Graph Theoretic Code (GTC) (associated with the graph defined by synaptic weight matrix). The set of all cuts in the graph of HNN are associated with the codewords of Graph Theoretic Code i.e. the cut edges and associated Bi-partite graph directly lead to the stable state (with '+1' components correspond to vertices on one side of cut and '–1' components correspond to vertices of the other side of cut i.e., {+1, –1} valued stable states are naturally associated with cuts in the graph of HAM). Thus, given the graph of HAM, we can enumerate all stable states using the generator matrix of graph theoretic code. Such an approach coupled with the synthesis problem naturally leads to the following issues.

5.1 Optimal Synthesis of Hopfield Associative Memory:

Maximize the minimum Hamming distance(d_{min}) between desired stable states (not necessarily orthogonal) which are not necessarily the eigen vectors of synaptic weight matrix, W. If possible arrive at upper bound on d_{min}.
Maximize the number of desired stable states with simultaneous maximization of d_{min}.
Note:

a) Stable states naturally define the bipartite subgraphs of the graph associated with HAM or HNN. Relaxing the earlier assumption, we denote the "Desired Stable States" (not necessarily the orthogonal eigen vectors) as those set of stable states (corresponding to cuts) with good distance property(d_{min}) and the number of them.

b) In the case of "even" or "odd" dimension(i.e., N), by choosing the W matrix to be singular(i.e., through suitable choice of null space of W), {+1, –1} vectors lying in the null space can be forced to be in the domain of attraction of $\{+\bar{e} \text{ or } -\bar{e}\}$ for NOISE IMMUNITY purposes. Effectively all such corners of hypercube are ensured to converge to a single stable state $\{+\bar{e} \text{ or } -\bar{e}\}$.

6 Conclusion

In this research paper, "optimal" synthesis of HAM, when dimension N is even and threshold vector is non-zero, is discussed. When 'N' is odd, one desired stable state HAM is synthesised. Also, "optimal" synthesis when N is odd is formulate and discussed.

References

[1] **[BrR]** J. Bruck, VP Roychowdhury," On the number of Spurious Memories in the Hopfield Model (Neural Network)" IEEE Transactions on Information Theory 36(2), pp.393-397, 1990.

[2] **[RaM]** G. Rama Murthy and Moncef Gabbouj," On the design of Hopfield Neural Networks: Synthesis of Hopfield type associative memories" International Joint Conference on Neural Networks (IJCNN),2015.

[3] **[BrB]** J. Bruck and M. Blaum, "Neural Networks, Error correcting codes and polynomial over Binary n-cube" IEEE Transactions on Information Theory Vol. 35, September 1989.