1f Fluctuations on a Chain of Causal Relationships

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$1/f$ fluctuations on a chain of causal relationships

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Abstract : In order to discuss the $1/f$ problem, the statistics of branching processes of particles in a multiplicative medium are applied to generate a series of intervals of two successive particle-counts by a detector, which has a spectrum behaving like $1/f$ over seven decades of frequency. It is also discussed that the $1/f$ fluctuations being familiar around us are strongly connected with a chain of causal relationships in a natural phenomenon, which is a new approach in this field.

1. Introduction

Since the observation of the $1/f$ spectrum of shot noise by Johnson in 1925 [1], this kind of spectrum has been observed in a large number of phenomena. The aim of the present work is to apply the branching process model discussed in my old works [2,3,4] for generating a series whose spectrum is characterized by a $1/f$ distribution in a wide range of frequency and to point out that the $1/f$ phenomena are essentially connected with the observations of events on a complicated chain of causal relationships.

In the stochastic process such as the Markov process, an event in the process is really influenced by the event happened just before. The time series composed of these events has usually a $1/f^2$-like spectrum. In the present work, a medium, in which many particles exist and each single particle may branch into several particles and be absorbed, is considered, where the particle number at a moment is decided stochastically by that just before. If these particles are observed by a detector in the medium, the observed events may show another kind of stochastic behavior. This is the basic idea of my works [2,3,4] and is illustrated in Fig. 1.

The particle numbers at time t_1 and t_2 in Fig. 1 are seven and eleven, respectively. A particle-count may correlated with another count through the branching paths as shown by a, b and c in the figure. The detection d has no correlation with a, b and c, because it is on another branching chain different from that of a, b and c. The length of the path between the counts has statistical correlation with the physical time interval. The time interval, for example, between a and b is approximately equivalent to that between b and c, but the correlation between b and c may be far weaker than that between *a* and *b*, because the path between *b* and *c* is longer than that between *a* and *b*.

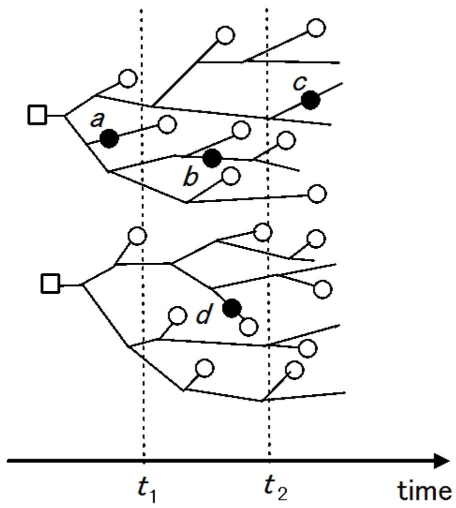

Figure 1. Chains of the branching processes. The square, circle and black spot represent a particle immigrated randomly in the medium, absorption and detection of a particle, respectively. The path of a particle is given by a full line.

2. Detection probability

We focus only on the case that exactly two particles are produced by a branching process, and suppose a medium in which a particle may be subjected to absorption, branching reaction and detection with the rates λ_a, λ_m and λ_d, respectively. In the previous works [2,3], only an absorption-type detector was considered in which a particle is absorbed by detection, however in this work, a non-absorption-type detector is also considered in which the particle detection has no influence on the particle number as shown in Fig. 1. The statistics of the particles and particle-counts in the medium can be obtained in close forms, and their detailed mathematical expressions are given in my previous works [2,3] in case the detector is an absorption-type. The results including the non-absorption-type case are shown in the following expressions obtained in a similar way to that given in my previous works.

We consider the probability $P_k(m,n,t)$ that *m* counts have been recorded by a detector during the time interval (0, *t*) and *n* particles are found in the medium at time $t > 0$ after *k* particles exist at $t = 0$ in the presence of random particle immigration with the rate *S*. It is rather complicated to obtain this probability for individual positive values of *m*, but when *m*=0, the probability $P_k(0,n,t)$ can be described in a close form as

$$P_k(0,n,t) = \sum_{i=0}^{n} K_k^{(0,n-i)} \cdot R_0^{(0,i)}, \tag{1}$$

where

$$K_k^{(0,n-j)} = \sum_{l=0}^{n-j} p(0,l,t) K_{k-1}^{(0,n-j-l)}$$
$$K_0^{(0,j)} = \delta_{j,0}, \tag{2}$$

and

$$p(0,l,t) = \begin{cases} \eta_0 \xi_0 V & (l=0) \\ \frac{(\eta_0-\xi_0)^2 e^{-\theta_0 t}}{(\eta_0-\xi_0 e^{-\theta_0 t})^2} & (l=1). \\ V \cdot p(0,l-1,t) & (l \geq 2) \end{cases} \tag{3}$$

Here

$$V = \frac{1-e^{-\theta_0 t}}{\eta_0-\xi_0 e^{-\theta_0 t}}. \tag{4}$$

The parameters θ_0, η_0 and ξ_0 in Eqs. (3) and (4) are given differently depending on the type of a detector of particles. When λ_c is the absorption rate of a particle in the medium except for that by a detector, the absorption rate λ_a defined before at the beginning of this section is given by

$$\lambda_a = \begin{cases} \lambda_c + \lambda_d & \text{(absorption type)} \\ \lambda_c & \text{(non − absorption type)} \end{cases}. \tag{5}$$

Defining the multiplication rate μ and the detection rate ε of a particle, respectively, as

$$\mu = \frac{\lambda_m}{\lambda_a} \text{ and } \varepsilon = \frac{\lambda_d}{\lambda_a}, \tag{6}$$

which are the number of particles appearing in the medium by branching processes and the number of detected particles, respectively, during the mean lifetime of a particle absorbed in the medium ($1/\lambda_a$), the parameters θ_0, η_0 and ξ_0 are expressed, respectively, as

$$\theta_0 = \begin{cases} \lambda_a \sqrt{(1-\mu)^2 + 4\varepsilon\mu} & \text{(absorption − type)} \\ \lambda_a \sqrt{(1+\varepsilon+\mu)^2 - 4\mu} & \text{(non − absorption type)} \end{cases}, \tag{7}$$

$$\eta_0 = \begin{cases} \frac{1}{2\mu}\left(1+\mu+\sqrt{(1-\mu)^2+4\varepsilon\mu}\right) & \text{(absorption − type)} \\ \frac{1}{2\mu}\left(1+\varepsilon+\mu+\sqrt{(1+\varepsilon+\mu)^2-4\mu}\right) & \text{(non − absorption type)} \end{cases} \tag{8}$$

and

$$\xi_0 = \begin{cases} \frac{1}{2\mu}\left(1+\mu-\sqrt{(1-\mu)^2+4\varepsilon\mu}\right) & \text{(absorption − type)} \\ \frac{1}{2\mu}\left(1+\varepsilon+\mu-\sqrt{(1+\varepsilon+\mu)^2-4\mu}\right) & \text{(non − absorption type)} \end{cases}. \tag{9}$$

The other factor $R_0^{(0,i)}$ in Eq. (1) is given by

$$R_0^{(0,i)} = \begin{cases} \exp\left[(\xi_0 - 1)St + \frac{S}{\mu\lambda_a}\ln\frac{\eta_0 - \xi_0}{\eta_0 - \xi_0 e^{-\theta_0 t}}\right] & (i = 0) \\ \frac{\frac{S}{\mu\lambda_a} + i - 1}{i} \cdot V \cdot R_0^{(0,i-1)} & (i \geq 1) \end{cases}. \tag{10}$$

The probability that some counts during the time interval (0, *t*) are recorded, i.e., $m > 0$, and *n* particles are found in the medium at $t > 0$ after we had *k* particles at $t = 0$ is given by

$$\sum_{m=1}^{\infty} P_k(m, n, t) = P_k(n, t) - P_k(0, n, t). \tag{11}$$

where $P_k(n, t)$ is the probability that *n* particles are found in the medium at time $t > 0$ after we had *k* particles at $t = 0$ and is given by

$$P_k\,(n, t) = \sum_{i=0}^{n} K_k^{(n-i)} Q_0^{(i)}. \tag{12}$$

Here

$$K_k^{(n-i)} = \sum_{l=0}^{n-i} p(l, t) K_{k-1}^{(n-i-l)} \tag{13}$$

$$K_0^{(i)} = \delta_{i,0}.$$

Defining the parameter α as

$$\alpha = \lambda_a - \lambda_m = (1 - \mu)\lambda_a, \tag{14}$$

the function $p(l, t)$ in Eq. (13) is described as

$$p(l, t) = \begin{cases} W_1 & (l = 0) \\ \left(\frac{1 - \mu}{1 - \mu e^{-\alpha t}}\right)^2 e^{-\alpha t} & (l = 1) \\ \mu W_1 \cdot p(l - 1, t) & (l \geq 2) \end{cases} \tag{15}$$

in case $\mu \neq 1$ (subcritical case), and

$$p(l, t) = \begin{cases} W_2 & (l = 0) \\ \left(\frac{W_2}{\lambda_a t}\right)^2 & (l = 1) \\ W_2 \cdot p(l - 1, t) & (l \geq 2) \end{cases} \tag{16}$$

in case μ=1 (critical case). Here, the functions W_1 and W_2 are given, respectively, by

$$W_1 = \frac{1 - e^{-\alpha t}}{1 - \mu e^{-\alpha t}}, \tag{17}$$

and

$$W_2 = \frac{\lambda_a t}{\lambda_a t + 1}. \tag{18}$$

The other factor in Eq. (12) $Q_0^{(i)}$ is obtained as, when $\mu \neq 1$,

$$Q_0^{(i)} = \begin{cases} \exp\left[\frac{S}{\mu\lambda_a}\ln\frac{1-\mu}{1-\mu e^{-\alpha t}}\right] & (i=0) \\ \frac{\frac{S}{\mu\lambda_a}+i-1}{i}\mu W_1 \cdot Q_0^{(i-1)} & (i \geq 1) \end{cases}, \tag{19}$$

and, when μ=1,

$$Q_0^{(i)} = \begin{cases} \exp\left[-\frac{S}{\lambda_a}\ln(1+\lambda_a t)\right] & (i=0) \\ \frac{\frac{S}{\lambda_a}+i-1}{i} W_2 Q_0^{(i-1)} & (i \geq 1) \end{cases}. \tag{20}$$

When the left side in Eq. (11) is much smaller than $P_k(0,n,t)$ due to a very short time interval, the probability recording more than two counts may be negligible and the following relation holds approximately.

$$P_k(1,n,t) \doteqdot P_k(n,t) - P_k(0,n,t). \tag{21}$$

3. Computer simulations

Whether particle detection has occurred or not in a very short time interval was decided successively by using the Monte Carlo method with the probabilities given by Eqs. (1), (12) and (21) with a non-absorption-type detector, from which a series formed by the time intervals between two successive counts (detection series) was obtained. In order to avoid a possibility that the particle number increases to infinity or dying out, the rate of random particle immigration was chosen to be $S = \alpha N_0$ when the multiplication rate $\mu < 1$ (subcritical case), considering the mean number of particles N_0 at $t \to \infty$ is S/α [2], and $S = 0$ when $\mu = 1$ (critical case). If $S > 0$ in a critical system, the particle number will increase with time and diverge eventually to infinity.

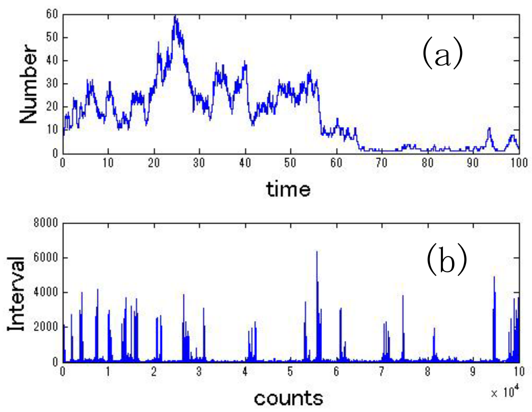

Figure 2. The fluctuations of (a) the particle series in case $\mu = 1$ and (b) the detection series in case $\mu = 1$ and $\varepsilon = 1$.

A very large number of existing particles takes unreasonably long time to process on a computer, and the number of particles should be set a limit. In the present simulations, the maximum number of particles was set a limit to 5000, and the mean number of particles N_0 was fixed to 100 when $\mu < 1$. The short time interval t in Eq. (21) was fixed to 5×10^{-6} measured in unit of $1/\lambda_a$ (mean lifetime of an absorbed particle) in all the simulations with the branching process model in the present work. The fluctuations of a part of the detection series are shown in Fig. 2 in comparison with those of the series for the number of particles existing in the medium (particle series) simulated with the probability given by Eq. (12). The detection series has a much more intermittent property than the other.

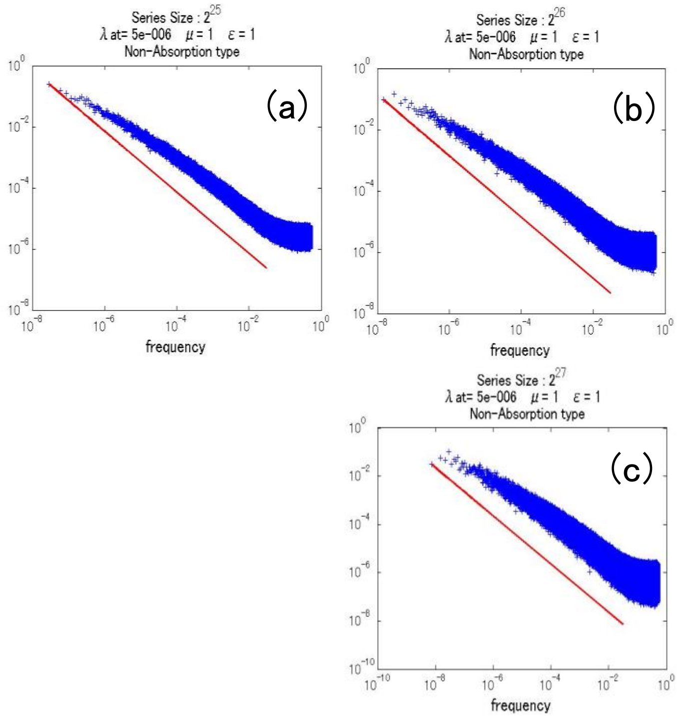

Figure 3. The spectra of the detection series with a non-absorption-type detector in case $\mu = 1$ and $\varepsilon = 1$. The series sizes are (a) 2^{25}, (b) 2^{26} and (c) 2^{27}, respectively. The straight lines give the $1/f$ behavior.

The fast Fourier transformation (FFT) results for several sizes of count series in case $\mu = 1$ and the detection rate $\varepsilon = 1$ are shown in Fig. 3. When the series size is $2^{25} \cong 3.4 \times 10^7$, the spectrum behaves like $1/f$ over seven decades of frequency, here the frequency is related to particle-counts and not to time. The spectrum for the series size over $2^{26} \cong 6.7 \times 10^7$ begins to deviate from the $1/f$ distribution in a low-frequency range. This deviation becomes more striking as the series size

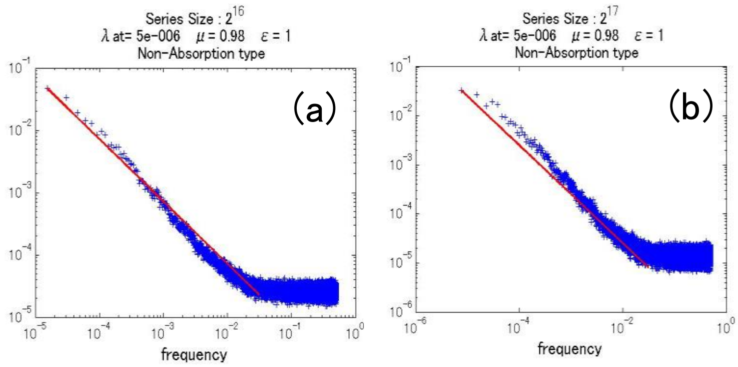

Figure 4. The spectra of the detection series with a non-absorption-type detector in case $\mu = 0.98$ and $\varepsilon = 1$. The series sizes are (a) 2^{16} and (b) 2^{17}, respectively. The straight lines give the $1/f$ behavior.

is longer and the spectral behavior converges to a finite value in a low-frequency range. In all the cases, the spectral behavior converges to a finite value in a high-frequency range. When $\varepsilon < 1$, it was examined by a similar way to the above that the deviation of the spectrum from the $1/f$ behavior in a low-frequency range starts at a shorter series size. The FFT results in case $\mu = 0.98$ and $\varepsilon = 1$ are shown in Fig. 4 where all the conditions except for μ are the same as those in case $\mu = 1$. When the series size is $2^{16} \cong 6.5 \times 10^4$, the spectrum behaves like $1/f$ for more than three decades of frequency, but the that at the series size of $2^{17} \cong 1.3 \times 10^5$ begins to deviate from the $1/f$ distribution in a low-frequency range. In this case, the frequency range where the spectrum behaves like $1/f$ is much narrower than the case of $\mu = 1$.

The longest series sizes behaving like $1/f$ have been obtained at various values of μ with an absorption-type detector as well as a non-absorption-type detector as shown in Fig. 5. The series size becomes sharply longer when $\mu > 0.98$. In cases $\mu = 0.90$ and 0.95, the series sizes behaving like $1/f$ are shorter when the particles are observed with an absorption-type detector compared to those with a non-absorption-type detector, but there is no clear difference between the results with these two types of detector at $\mu > 0.98$.

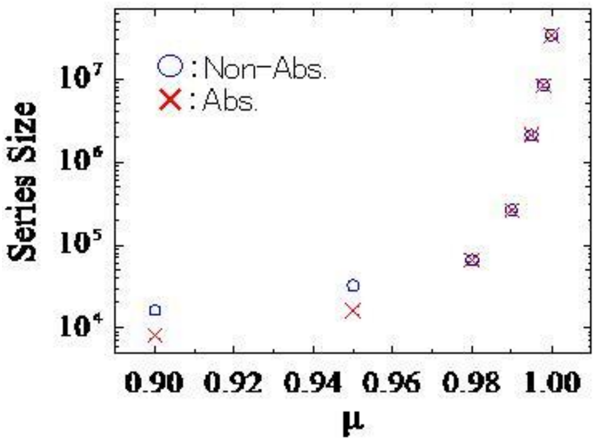

Figure 5. The longest series sizes behaving like $1/f$ at various values of μ. The circles and crosses are the results for the non-absorption-type and absorption-type detectors, respectively.

4. Discussions

From the simulations described above, it comes to light that a detection series formed by the time intervals between successive particle-counts in a medium where a particle may branch into two particles has a spectrum characterized by a $1/f$ distribution and the frequency range behaving like $1/f$ depends strongly on the value of the multiplication rate μ. While this frequency range is very wide when $\mu = 1$, this range becomes sharply narrower with decreasing μ, as shown in Figs. 3,4 and 5. The question, then, arises as to why the series size behaving like $1/f$ becomes so sharply shorter with decreasing μ, i.e., the series size changes from about 3.3×10^7 to 6.5×10^4 while the value of μ decreases only 0.02, from 1 to 0.98, as seen in Fig. 5.

The spectra of particle series in cases $\mu = 1$, 0.99, 0.98 and 0.9 are given in Fig. 6. When $\mu = 1$, the spectrum behaves like $1/f^2$ at the series size of 2^{28} and starts to deviate from the $1/f^2$

distribution in a low frequency range at the series size of 2^{29}. This situation is nearly kept even when $\mu = 0.99$, although the spectrum deviates slightly from the $1/f^2$ distribution at the series size of 2^{28}. On the other hand, when $\mu = 0.98$, this situation is realized only at a shorter series size, and at a much shorter series size when $\mu = 0.9$.

As can be seen in Fig. 6, the sizes of the particle series behaving like $1/f^2$ at $\mu = 0.98$ and 0.9 are, respectively, four times and sixteen times shorter compared with the case of $\mu = 0.99$. On the other hand, the particle series at $\mu = 0.99$ has a similar spectral characteristic to the case of $\mu = 1$, but the size of it's detection series behaving like $1/f$ is more than 100 times shorter compared with the case of $\mu = 1$, as seen in Fig. 5.

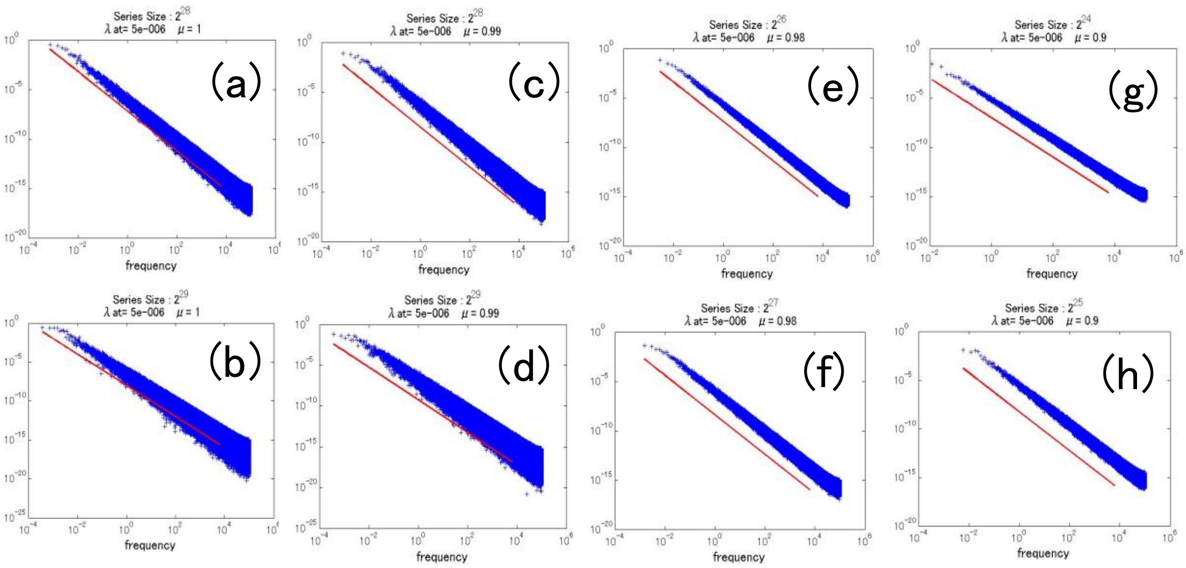

Figure 6. The spectra of the particle series. The multiplication rate μ and the series size in the figure are, respectively, (a) $\mu = 1$ and 2^{28}, (b) $\mu = 1$ and 2^{29}, (c) $\mu = 0.99$ and 2^{28}, (d) $\mu = 0.99$ and 2^{29}, (e) $\mu = 0.98$ and 2^{26}, (f) $\mu = 0.98$ and 2^{27}, (g) $\mu = 0.9$ and 2^{24}, and (h) $\mu = 0.9$ and 2^{25}. The straight lines give the $1/f^2$ behavior.

Although the spectral characteristics of the particle series at $\mu = 0.99$ and $\mu = 1$ are quite similar to each other, why are their detection series sizes behaving like $1/f$ so different? As written before, the random immigration $S = 0$ when $\mu = 1$, and so all the particles in the medium are bred from a single original particle and are on the same branching chain as can be understood in Fig. 1. This means that every single particle in the medium has a correlation with all the other particles and, therefore, each single count has a correlation with the other counts. On the other hand, as $S \neq 0$ when $\mu < 1$, some detection-counts have no correlation with others like as the detection *d* in Fig. 1. The smaller the multiplication rate μ is, the more particles are immigrated in the medium randomly in order to avoid the possibility of the particle number dying out and the more detection-counts having no correlation with others increase in number. These considerations suggest strongly that a correlation between particles on a single branching chain plays an important role in the spectral characteristics of a detection series behaving like $1/f$ in a wide range of frequency.

The effect of the correlation between particles has been examined also by another kind of simulations, where it was assumed that the particle number n_0 at $t = 0$ will change, at later time t, to another number n_t distributing normally around n_0 with a standard deviation $\sqrt{n_0}/N_{fix}$, where N_{fix} is a parameter describing the width of the deviation. In the simulations, the maximum number of particles was also set a limit to 5000 and the detection probability of an individual particle was set to 0.01/5000 during the time interval (0, t), which gives the maximum detection probability 0.01 when the particle number is 5000. In the simulations with the branching process model we have a definite unit of time such as the mean lifetime of a particle absorbed in the medium, but there is no definite unit of time now, and so, the unit of time was supposed to be the time interval (0, t). An example of the simulations is given in Fig. 7 in case $N_{fix} = 30$.

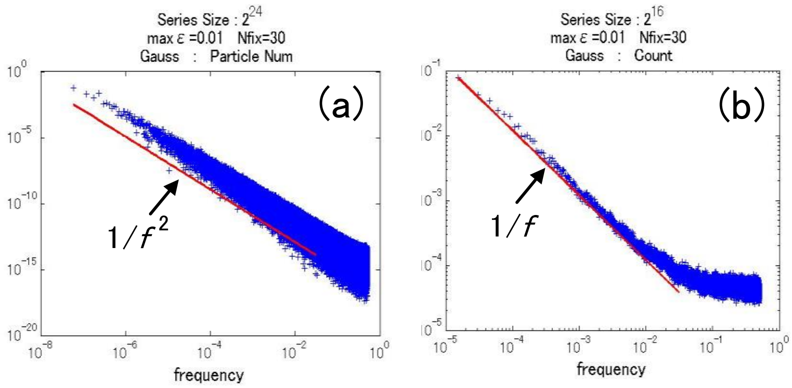

Figure 7. The spectra of (a) the particle series and (b) the detection series simulated with the model without assuming the branching processes. The straight lines give the $1/f^2$ and $1/f$ behavior in (a) and (b), respectively. When the series size is longer than 2^{24} in case of the particle series or 2^{16} in case of the detection series, the spectrum begins to deviate from the $1/f^2$ or $1/f$ distribution, respectively, in the low-frequency range.

In the present simulations, the particle number at a moment is really influenced just before, and the particle series has a spectrum characterized by a $1/f^2$ distribution. The correlation, however, between the particles in this model is uncertain. This comes out that the frequency ranges behaving like $1/f^2$ and $1/f$ are not sufficiently wide in the spectra of the particle series and the detection series, respectively, as shown in Fig. 7.

5. Conclusions

The above discussions reveal that the relationship between particles is definitely important to realize a $1/f$ spectrum in a wide range of frequency. When $\mu = 1$, all the particles in the medium are related with the other particles on a single branching chain and, therefore, there is no particle which is absolutely independent of the other particles. This is the reason why the spectrum of a detection series behaves like $1/f$ in a really wide range of frequency.

This result obtained in the present research gives us an instructive suggestion about the $1/f$ fluctuations frequently observed in natural phenomena. The global environment is affected by the sunlight and solar heat, the heat of the earth's interior, rotation and revolution of the earth, and so on. The global condition at a moment is actually influenced by the condition just before, and therefore its change with time is moderate as shown in Fig. 2(a). On the other hand, the local condition of the environment is so complicated, and is different from place to place. A local event at a place is an all-inclusive result of innumerable tiny events being correlative with each other, and is influenced by many other events sometimes directly and sometimes indirectly, i.e., influenced directly by other events just beside and/or indirectly by other events far from this place through a complicated chain of causal relationships. It is possible to explain this situation by an example of a breeze in a forest. This blowing is, as a whole, a stream of the air in one direction. However, the local conditions in the stream are so complicated. It blows gently somewhere, while strongly at another place, because of a complicated configuration of trees, a rolling ground, the inhomogeneous temperature distribution and the other different reasons. A blowing at a place and a moment may be influenced, directly or indirectly, by the other blowing at other places and moments. In this way, there is a complicated chain of causal relationships between innumerable tiny events, like as the particles on a branching chain as illustrated in Fig. 1. It is usually difficult to observe the whole of the events on the chain. We observe only a part of the events on a chain of causal relationships and, therefore, the relation between the observed events is complicated and difficult to understand clearly. These above considerations suggest that our usual observation of a part of the events on a chain of causal relationships may be one of the reason why we often find the $1/f$ phenomena around us. These situations remind us of the idea of interdependent co-arising in Buddhism.

In conclusion, the simulations with the branching process model reveal that a spectrum of a series formed by the time intervals between two successive particle-counts by a detector behaves like $1/f$ in a wide range of frequency when the particles in the medium are connected with each other on a single branching chain. On the analogy of the above results, it is supposed further that the observation of a part of events on a chain of causal relationships composing a natural phenomenon may come to a $1/f$ spectrum which is really familiar around us.

6. References

[1] J. B. Johnson, *Phys. Rev*. **26,** 71 (1925).

[2] T. Kobayashi, *J. Phys. A: Math, Gen.* **21,** 3723 (1988).

[3] T. Kobayashi, *J. Phys. A: Math, Gen.* 22, L931 (1989).

[4] T. Kobayashi, *Proc. Int. Conf. on Noise and Fluctuations (Neuchatel, Switzerland)* p 278, (2019).