# Globally Stable Synchronization Conditions in Total Diffusive Linear Bidirectional Coupling Between Continuous Dynamical Systems and Partial Replacement (Rössler and Lorenz)

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**Abstract**

In order to obtain asymptotical synchronization, we combine diffusive linear bidirectional coupling with partial replacement on the nonlinear terms of the second system, a coupling version that was less explored. All these bidirectional coupling schemes are established between Lorenz systems or Rössler systems with chaotic behavior/with control parameters that lead to chaotic behavior**.**

The sufficient conditions of global stable synchronization are obtained from a different approach of the Lyapynov direct method for the transversal system. In one coupling we apply a result based on classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite, where $\mathbf{A}$ is the matrix characterizing the transversal system. In the remaining couplings the sufficient conditions are based on (the) increase/accretion of derivative **(quero dizer majoração da derivada)** of an appropriate Lyapunov function assuming yet the limitation of certain variables. In fact, the effectiveness of a coupling between systems with equal dimension follows of the analysis of the synchronization error $\mathbf{e}(t)$ and, if the system variables can be bounded by positive constants, the derivative of an appropriate Lyapunov function can be increased.**(quero dizer majorada)** as required by the Lyapynov direct method.

In what follows we will always consider two chaotic dynamical systems, since they are sufficient to study the essential in the proposed coupling schemes. Our motivation for researching chaos synchronization methods is to explore their practical application in various scientific areas, such as physics, biology or economics.

## 1 Introduction

The ability of nonlinear oscillators to synchronize with each other is a basis for the explanation of many processes of nature. Therefore, chaos synchronization is thus a robust property expected to hold in mademan devices and plays a significant role in science. However, the possibility of two (or more) chaotic systems oscillate in a coherent and synchronized way is not an obvious phenomenon, since it is not possible to reproduce exactly the initial conditions andinfinitesimal perturbations to them/the initial conditions lead to divergence of nearby starting orbits. Contrary to expectation, when ensembles of chaotic oscillators are coupled, the attractive effect of a suitable coupling can counterbalance the trend of the trajectories to diverge. In many cases there are (coupling) parameters that control the strength of coupling between the systems, and the stability results of synchronous chaotic state depend on them.

Coupled dynamical systems are constructed from simple, low-dimensional dynamical systems and form new and more complex organizations. The chaotic dynamics introduces new degrees of freedom in ensembles of coupled systems. However, when two or more chaotic oscillators are coupled and synchronization is achieved, in general the number of dynamic degrees of freedom for the coupled system effectively decreases.

**Asymptotical synchronization.** Let $X$ be a compact subset of $\mathbb{R}^m$ with $m \geq 3$ and consider (two) identical $m$-dimensional dynamical systems $S_1$ and $S_2$ defined on $X$ by the nonlinear autonomous ordinary differential equations (ODE) $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$ and $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$, respectively, where $\mathbf{a}$ is a vector of real control parameters.

Let $\mathbf{u}_1(0)$ and $\mathbf{u}_2(0)$ be (some) initial conditions for which, at certain value of $\mathbf{a}$, $S_1$ and $S_2$ evolve to an asymptotically stable chaotic attractor $\mathcal{A}$. The solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ of the systems, starting at $\mathbf{u}_1(0) \neq \mathbf{u}_2(0)$ in the attraction basin $\mathcal{B}(\mathcal{A})$, are/represent independent trajectories in $\mathcal{A}$ after a period time of transient motion. This evolution is characterized by a positive Lyapunov exponent. Dynamical systems $S_1$ and $S_2$ are *asymptotically synchronized* if

$$\lim_{t \to +\infty} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| = 0, \tag{1}$$

and are *fully synchronized* if $\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| = 0$ for $t > t_{sync}$, with $t_{sync} \in \mathbb{R}$ called *synchronization time*.

The evolution of the difference $\mathbf{e}(t) = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ between nearby starting orbits is described by

$$\dot{\mathbf{e}}\,(t) = \dot{\mathbf{u}}_2\,(t) - \dot{\mathbf{u}}_1\,(t) = \mathbf{f}(\mathbf{u}_2\,(t)\,;\mathbf{a}) - \mathbf{f}(\mathbf{u}_1\,(t)\,;\mathbf{a}). \tag{2}$$

In case of asymptotical synchronization, this difference is the synchronization error and the system (2) is designated as transversal system (or error system). By (1), $S_1$ and $S_2$ achieve asymptotical synchronization if the transversal system (2) has an asymptotically stable equilibrium point at $\mathbf{e}\,(t) = \mathbf{0}$.

When asymptotical synchronization is achieved, the dynamics of $\mathbf{u}_1\,(t)$ and $\mathbf{u}_2\,(t)$ in $\mathcal{A}$, on the $2m$-dimensional phase space, are restricted to the $m$-dimensional smooth invariant manifold

$$\mathcal{M} \equiv \{(\mathbf{u}_1, \mathbf{u}_2) \in X \times X \mid \mathbf{u}_1 = \mathbf{u}_2\} \subset \mathbb{R}^{2m},$$

where occurs the synchronized dynamics defined by the symmetric synchronous chaotic state.

**Transversal stability of the coupled system.** The problem of synchronization can be understood as a problem of asymptotical stability of the chaotic attractor $\mathcal{A}$ (embedded in $\mathcal{M}$) in the $2m$-dimensional phase space of the coupled system (Fujisaka and Yamada [1], Pikovsky [2], Pecora and Carroll [3]).

It is necessary to distinguish between stability under tangent or transversal perturbations to (the) synchronization manifold $\mathcal{M}$. As stated by Pecora *et al.* [4], the limit (1) must be satisfied for all the initial conditions in a neighborhood of the equlibrium point $\mathbf{e}\,(t) = \mathbf{0}$. Since the system (2) characterizes the dynamics in the transversal direction to $\mathcal{M}$, it is necessary to analyze if small transversal perturbations to $\mathcal{M}$ are reduced or amplified by the evolution of $S_1$ and $S_2$. If they are reduced then $\mathcal{M}$ is transversely stable and the synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ is stable. So, the synchronization stability is designated as transversal stability.

Usually the following criteria are applied:

*(i)* Criterion based on the eigenvalues of (the) Jacobian matrix corresponding to the flow over $\mathcal{M}$, suggested by Fujisaka and Yamada ([1],[5]); it requires that the largest eigenvalue is negative for the early stable synchronization;

*(ii)* Criterion based on the construction and study of an appropriate Lyapunov function $L\,(\mathbf{e}\,(t))$ (Lyapunov direct method) for the vector field of transversal perturbations to $\mathcal{M}$, developed by He and Vaidya [6]; it requires that $L$ must be positive definite in a neighborhood of $\mathcal{M}$ $(L\,(\mathbf{e}\,(t)) \geq 0)$,

except in $\mathcal{M}$ where is null ($L(\mathbf{0}) = 0$), and its derivative is negative semi-definite ($\dot{L}(\mathbf{e}(t)) \leq 0$) and null in $\mathcal{M}$ ($\dot{L}(\mathbf{0}) = 0$);

*(iii)* Criterion based on the estimation of Lyapunov exponents, developed by Pecora and Carroll [3], which indicate if small transversal perturbations $e_i(t)$, *for* $1 \leq i \leq m$, decrease or not; it requires that the largest transversal Lyapunov exponent is negative.

The criterion *(ii)* allows to prove the following/next proposition about global asymptotical stability of transversal system defined by (2).

**Proposition 1** *Let* $\mathbf{A}$ *be the matrix characterizing the transversal system of a coupling between identical systems* $S_1$ *and* $S_2$*. If there is a constant* $\delta < 0$ *such that the symmetric matrix* $\mathbf{A}^T + \mathbf{A}$ *is negative definite and* $\mathbf{A}^T + \mathbf{A} \leq \delta\mathbf{I}$ *for any* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *in the phase space* $X$*, then the dynamics of the transversal system is globally stable and the systems* $S_1$ *and* $S_2$ *are in stable synchronization.*

**Proof.** Consider the Lyapunov function defined by $L(\mathbf{e}(t)) = [\mathbf{e}(t)]^T \cdot \mathbf{e}(t)$. Its derivative is given by

$$\frac{dL}{dt}(\mathbf{e}) = \frac{d\left(\mathbf{e}^T\right)}{dt} \cdot \mathbf{e} + \mathbf{e}^T \cdot \frac{d\mathbf{e}}{dt} = \mathbf{e}^T \cdot \mathbf{A}^T \cdot \mathbf{e} + \mathbf{e}^T \cdot \mathbf{A} \cdot \mathbf{e},$$

and verifies

$$\dot{L}(\mathbf{e}) = \mathbf{e}^T\left(\mathbf{A}^T + \mathbf{A}\right)\mathbf{e} \leq \delta\left(\mathbf{e}^\mathbf{T} \cdot \mathbf{I} \cdot \mathbf{e}\right) = \delta\left(\mathbf{e}^T \cdot \mathbf{e}\right) < 0$$

for all $\mathbf{e} \neq \mathbf{0}$. The Lyapunov direct method guaranties the global asymptotical stability of transversal system ∎

# 2 Coupling schemes between continuous chaotic dynamical systems/chaotic dynamical systems defined by ODE

## 2.1 Diffusive linear bidirectional coupling

According Fujisaka and Yamada ([1],[5]), a natural way to introduce a dissipative coupling between identical chaotic systems $S_1$ and $S_2$ is to add symmetric linear coupling terms to the expressions that define them. This coupling mechanism is designated by *diffusive linear coupling* and plays a central role in chaos control. By this bidirectional coupling between $S_1$ and $S_2$ it is defined the coupling system

$$\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a}) + \mathbf{D}_1(\mathbf{u}_2 - \mathbf{u}_1) \quad \wedge \quad \dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a}) + \mathbf{D}_2(\mathbf{u}_1 - \mathbf{u}_2), \qquad (3)$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are coupling diagonal matrices of order $m$ with diagonal elements $\rho_{1,i}$ and $\rho_{2,i}$ that are positive or zero, respectively.

If all the pairs $(\rho_{1,i}, \rho_{2,i})$ of corresponding diagonal elements are non-zero, that is, if $\rho_{1,i} \neq 0$ and/or $\rho_{2,i} \neq 0$ for $i = 1, \ldots, m$, the coupling is said *total*. If in the diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ there are pairs $(\rho_{1,i}, \rho_{2,i})$ of corresponding diagonal elements null, that is, if $\rho_{1,i} = 0$ e $\rho_{2,i} = 0$ for some $1 \leq i \leq m$, is carried out only a *partial coupling*. In this case it is not made the coupling of some equations of the coupling system (3).

**Stability of coupling system.** In total coupling the evolution of synchronization error $\mathbf{e} = \mathbf{u}_2 - \mathbf{u}_1$ is characterized by the transversal system

$$\|\dot{\mathbf{e}}(t)\| = \|\mathbf{f}(\mathbf{u}_2(t); \mathbf{a}) - \mathbf{f}(\mathbf{u}_1(t); \mathbf{a})\| - (d_1 + d_2) \cdot \|\mathbf{e}(t)\|,$$

where $d_1 + d_2$ denotes the sum of the diagonal elements $\rho_{1,i}$ and $\rho_{2,i}$ of matrices $\mathbf{D}_1$ and $\mathbf{D}_2$. The distance between the individual trajectories of (systems) $S_1$ and $S_2$ is given by equation

$$\|\mathbf{e}(t)\| = \delta_0 e^{\lambda_{\max} t},$$

where $\lambda_{\max}$ is the value of the largest positive Lyapunov exponent and $\delta_0 \geq 0$ is an infinitely small initial distance between the trajectories. By deriving it is obtained the relation

$$\|\mathbf{f}(\mathbf{u}_2(t); \mathbf{a}) - \mathbf{f}(\mathbf{u}_1(t); \mathbf{a})\| = \lambda_{\max} \delta_0 e^{\lambda_{\max} t}$$

and, given the equality

$$\delta_0 \|\mathbf{e}(t)\| = e^{-\lambda_{\max} t},$$

the equation

$$\|\dot{\mathbf{e}}(t)\| = \lambda_{\max} \delta_0 e^{\lambda_{\max} t} - (d_1 + d_2) \cdot \|\mathbf{e}(t)\|$$

can be written as an ordinary differential equation of separate variables whose solution is

$$\|\mathbf{e}(t)\| = \mathbf{e}_0 e^{\lambda_{\max} t} e^{-(d_1 + d_2)t} = \mathbf{e}_0 e^{[\lambda_{\max} - (d_1 + d_2)]t}, \tag{4}$$

where $\mathbf{e}_0$ is the initial difference between the trajectories. One may also consider $\mathbf{f}(\mathbf{u}_2; \mathbf{a}) = \mathbf{f}(\mathbf{u}_1; \mathbf{a}) = 0$ in (3) which results in the system

$$\dot{\mathbf{u}}_1 = \mathbf{D}_1 \mathbf{e} \quad \wedge \quad \dot{\mathbf{e}}_2 = -(\mathbf{D}_1 + \mathbf{D}_2)\, \mathbf{e}$$

whose solution is given by $\|\mathbf{e}(t)\| = \mathbf{e}_0 e^{-(d_1 + d_2)t}$.

From/Given (4) (it can be concluded that) the synchronization error $\mathbf{e}$ results from two independent properties. First/On the one hand, the exponential divergence of nearby trajectories by a ratio $\lambda_{\max} t$ proportional to the positive Lyapunov exponent. Moreover/On the other hand, the exponential convergence resulting from the coupling terms $\mathbf{D}_1 (\mathbf{u}_2 - \mathbf{u}_1)$ and $\mathbf{D}_2 (\mathbf{u}_1 - \mathbf{u}_2)$ by a ratio $-(d_1 + d_2)\, t$ proportional to the sum of coupling coefficients. Making $\mathbf{f}(\mathbf{u}_2; \mathbf{a}) = \mathbf{f}(\mathbf{u}_1; \mathbf{a}) = 0$, the transversal system for the evolution of $\mathbf{e}$ associated to this convergence takes the form

$$\dot{\mathbf{e}}(t) = -(d_1 + d_2) \cdot \mathbf{e}(t) .$$

While the exponential convergence property acts/performs over all the phase space $X$, the first one acts only in a neighborhood of the synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ where linear effects are dominant. As such, the product of both exponential factors in (4) only takes place near (the manifold synchronization) $\mathcal{M}$.

According to (4), the full synchronization in coupled system (3) occurs if the inequality

$$d_1 + d_2 > \lambda_{\max} \tag{5}$$

is valid. The full synchronization condition (5) shows that there is a linear dependence between the maximal Lyapunov exponent of systems and the elements of diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ (of coupling between them).

The Stefański's study in [7] shows that the exponential divergence and convergence properties in total coupling allow to estimate the largest Lyapunov exponent of any dynamical system chaotic. This possibility is especially useful in non-smooth systems, where the estimation of Lyapunov exponents is not direct [7].

The synchronization condition does not ensure a fully synchronized state when only a partial coupling is made. Indeed, in the absence of coupling of one or more equations of the coupled system (3), the effect of coupling terms $\mathbf{D}_1 (\mathbf{u}_2 - \mathbf{u}_1)$ and $\mathbf{D}_2 (\mathbf{u}_1 - \mathbf{u}_2)$ may not be as regular (in the synchronization) as in total coupling. Stefański and Kapitaniak [8] introducing a coupling coefficient $\zeta > 0$ which allows to consider the synchronization condition $d_1 + d_2 > \zeta \lambda_{\max}$ that evaluates the effectiveness level of partial coupling. This condition is a generalization of (5), since in total coupling the value of $\zeta$ increases/comes to 1.

The analytical determination of coupling coefficient $\zeta$ (in partial coupling) is difficult and may even be impossible. However, it may be estimated in the course of numerical experiments according to (the relation)

$\zeta = \rho_{\min}/\lambda_{\max}$, where $\rho_{\min}$ denotes the minimum value of the sum $d_1 + d_2$ of diagonal elements $\rho_{1,i}$ and $\rho_{2,i}$ for which occurs stable synchronous movement. In many systems the value of $\zeta$ is independent of the initial conditions (taken) [8].

When the diagonal elements $\rho_{1,i}$ and $\rho_{2,i}$ verify $\rho_{1,i} = \rho_{2,i} = \rho_i$, the coupling proposed in (3) leads to the coupled system

$$\begin{cases} \dot{u}_i = f_i\left(u_1, ..., u_m; \mathbf{a}\right) + \rho_i\left(u_i' - u_i\right) \\ \dot{u}_i' = f_i\left(u_1', ..., u_m'; \mathbf{a}\right) + \rho_i\left(u_i - u_i'\right) \end{cases}, \quad 1 \leq i \leq m.$$

The symmetrical chaotic synchronous solution $u_i\left(t\right) = u_i'\left(t\right) \equiv U_i\left(t\right)$ of this system, corresponding to the synchronous state $\mathbf{u}_1 = \mathbf{u}_2$, and the study of their stability to small transversal perturbations $e_i\left(t\right) = u_i'(t) - u_i(t)$, for $1 \leq i \leq m$, requires the linearized equation

$$\dot{e}_i = \frac{\partial f_i}{\partial x_j}\left(\mathbf{U}\left(t\right)\right) e_j - 2\rho_i e_i. \tag{6}$$

(The) Solutions of transversal system (6) increases exponentially as $t \to +\infty$. As $m$ is the dimension of this system, then there are $m$ transversal Lyapunov exponents and the largest one, $\lambda_{\max}^{\perp}$, determines the stability of perturbations. Since the expressions $2\rho_i e_i$ in (6) depend on the coordinates $\rho_i$ of parameter vector $\boldsymbol{\rho}$, the maximal exponent $\lambda_{\max}^{\perp}$ depends on these coordinates and the condition $\lambda_{\max}^{\perp}\left(\rho_i\right) < 0$ defines the synchronization region.

**Particular case.** Consider the particular case in which the diagonal elements have the same value, that is $\rho_1 = \cdots = \rho_m \equiv \rho > 0$,

$$\begin{cases} \dot{u}_i = f_i\left(u_1, ..., u_m; \mathbf{a}\right) + \rho\left(u_i' - u_i\right) \\ \dot{u}_i' = f_i\left(u_1', ..., u_m'; \mathbf{a}\right) + \rho\left(u_i - u_i'\right) \end{cases}, \quad 1 \leq i \leq m. \tag{7}$$

In order to analyze the transversal stability of the synchronous state $\mathbf{u}_1 = \mathbf{u}_2$, consider new variables

$$\mathbf{U}(t) = \frac{1}{2}\left[\mathbf{u}_1(t) + \mathbf{u}_2(t)\right] \quad \text{e} \quad \mathbf{V}(t) = \frac{1}{2}\left[\mathbf{u}_1(t) - \mathbf{u}_2(t)\right]$$

in (7). Variable $\mathbf{V}(t)$ describes the transversal evolution to invariant manifold $\mathcal{M}$ while, in the limit of non-transversal movement, $\mathbf{U}(t)$ describes the evolution in $\mathcal{M}$. With this change of variables, the coupled system (7) can be rewritten as

$$\begin{cases} \dot{\mathbf{U}} = \dfrac{1}{2}\left[\mathbf{f}(\mathbf{U}+\mathbf{V}) + \mathbf{f}(\mathbf{U}-\mathbf{V})\right] \\ \\ \dot{\mathbf{V}} = \dfrac{1}{2}\left[\mathbf{f}(\mathbf{U}+\mathbf{V}) - \mathbf{f}(\mathbf{U}-\mathbf{V})\right] - 2\rho\mathbf{V} \end{cases}. \tag{8}$$

To analyze the stability of the transversal subspace to $\mathcal{M}$ it is equivalent to show that the transversal dynamical system in the variable $\mathbf{V}(t)$ has an asymptotically stable equilibrium point at the origin. The Lyapunov exponents spectrum of equation (8) can be split into two subsets: $\boldsymbol{\lambda}^{\parallel}$ constituted by the tangential Lyapunov exponents associated to the evolution of $\mathbf{U}(t)$, which describes the dynamics in $\mathcal{M}$ or close to it, and $\boldsymbol{\lambda}^{\perp}$ composed of the transversal Lyapunov exponents which characterize the evolution of small perturbations transverse to this manifold. By criterion *(iii)*, the chaotic attractor $\mathcal{A}$ is stable if all the transversal Lyapunov exponents are negative. Let $D\mathbf{f}(\mathbf{U})$ be the Jacobian matrix corresponding to the linearization around the equilibrium point $\mathbf{u}_1(t) = \mathbf{u}_2(t) = \mathbf{U}(t)$. If the largest transversal Lyapunov exponent $\lambda^{\perp}_{\max}(\rho)$ corresponding to $D\mathbf{f}(\mathbf{U})$ is negative, then any transversal perturbation to $\mathcal{M}$ is damped and the synchronous state $\mathbf{u}_1 = \mathbf{u}_2$ is stable.

If, in addition to $\mathbf{V}(t) \to 0$ as $t \to +\infty$, it is also verified the condition

$$\frac{d\,\|\mathbf{V}(t)\|}{dt} < 0,$$

then takes place a special case of synchronization, called *monotonic synchronization.* In the case of monotonic synchronization for all initial values in the neighborhood of $\mathbf{V}(t) = \mathbf{0}$, the chaotic attractor $\mathcal{A}$ is called *monotonically asymptotically stable.*

Are presented in Table 1 frm a topological point of view in terms of coupling parameter $\rho$, the stability transitions of $\mathcal{A}$ embedded in $\mathcal{M}$ considering the bifurcations of the coupled system.

| **Parameter $\rho$** | **Chaotic attractor $\mathcal{A}$** |
|---|---|
| $\rho < \rho_0$ | Repulsive/Repeller chaotic saddle |
| $\rho_0 < \rho < \rho'''$ | with on-off intermittency (chaos-hipercaos) |
| $\rho''' < \rho < \rho''$ | with locally riddled basin |
| $\rho'' < \rho < \rho'$ | asymptotically stable |
| $\rho > \rho'$ | monotonically asymptotically stable |

Table 1: Behavior of chaotic attractor $\mathcal{A}$ as a function of/depernding on $\rho$.

The stronger phase of asymptotic stability occurs for/when $\rho > \rho'$ but, if

$$\rho > \rho'' \equiv \frac{1}{2}\left(\lambda_{\max}^{\|}(\rho)\right),$$

where $\lambda_{\max}^{\|}(\rho)$ is the maximal tangential Lyapunov expoent of $\mathcal{A}$, it is achieved synchronization for all the initial conditions in the neighborhood of $\mathcal{A}$ [5]. At $\rho = \rho''$ occurs a bifurcation in which $\mathcal{A}$ loses its asymptotic stability. As evidenced by Alexander *et al.* [9], Sommerer and Ott [10] and Ott *et al.* [11], when $\rho < \rho''$ the attractor $\mathcal{A}$ is stable but there may be a distance $\delta > 0$ from $\mathcal{A}$ such that, for each point $\mathbf{u}_1 \in \mathcal{B}(\mathcal{A})$, any arbitrarily small ball centered at $\mathbf{u}_1$ contains a set of points of positive measure whose orbits exceed $\delta$. Given a typical trajectory of (8), although/though all the transverse Lyapunov exponents are negative, there are initial conditions (dense) in $\mathcal{A}$ for which one of them is positive. Taking values of $\rho$ even/still smaller, the system (8) undergoes a blowout bifurcation in a certain value $\rho'''$ that characterizes the transition from chaos to hipercaos ([1],[2],[3],[12],[13]). According to Nusse and Yorke [14], when $\rho < \rho'''$ there is a neighborhood $W$ of $\mathcal{A}$ such that the set $\mathcal{B}(\mathcal{A}) \cap W$ contains $\mathcal{A}$ but its Lebesgue measure is zero. A typical trajectory spends some time in the neighborhood of $\mathcal{A}$ but occasionally bursts away from it/him. (The) Maximal transversal Lyapunov expoent of $\mathcal{A}$, $\lambda_{\max}^{\perp}(\rho)$, is always positive but with low value. However the finite time fluctuations can allow that all the transient Lyapunov exponents are negative in some time periods in which the orbit is attracted to $\mathcal{M}$. (The) Chaotic attractor $\mathcal{A}$ thus becomes a chaotic saddle. For $\rho$ below a certain value $\rho_0$, $\lambda_{\max}^{\perp}(\rho)$ is large enough allowing the evolution to a distinct attractor.

### 2.2 Unidirectional coupling by partial replacement

**(será melhor retira a 1$^a$ parte?)** Consider an (arbitrary) decomposition $\mathbf{u}_1 = (\mathbf{x}_1, \mathbf{y}_1)$ of the variable $\mathbf{u}_1$ into two subsystems

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}), \tag{9}$$

with variables $\mathbf{x}_1 = (u_1, \ldots, u_k)$ and $\mathbf{y}_1 = (u_{k+1}, \ldots, u_m)$, respectively, for $1 \leq k \leq m$. Since $\mathbf{f}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_m(\mathbf{u}_1; \mathbf{a}))$, the vector fields $\mathbf{g}$ and $\mathbf{h}$ are defined by the component functions of the vector field $\mathbf{f}$ as

$$\mathbf{g}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_k(\mathbf{u}_1; \mathbf{a}))$$

and

$$\mathbf{h}(\mathbf{u}_1;\mathbf{a}) = (f_{k+1}(\mathbf{u}_1;\mathbf{a}), \ldots, f_m(\mathbf{u}_1;\mathbf{a})).$$

They are respectively taken independent initial conditions $\mathbf{x}_1(0)$ and $\mathbf{y}_1(0)$ in the subsystems in (9). Let $\dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$ be a subsystem identical to $\dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ with the variable $\mathbf{x}_1$ replaced by its corresponding $\mathbf{x}_2$,

$$\mathbf{x}_2 = \mathbf{x}_1 \qquad \text{and} \qquad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}).$$

So, the equations

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}), \tag{10}$$

with $\mathbf{y}_2(0) \neq \mathbf{y}_1(0)$, defined a dynamical system $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$ which shares some of the variables with the system $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$. Pecora e Carroll [3] formalized this unidirectional coupling between the systems (9) and (10) through the variable $\mathbf{x}_1$, $\dot{\mathbf{u}}_2 = \mathbf{f}_{x_2 \to x_1}(\mathbf{u}_2; \mathbf{a}) = \mathbf{f}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$, where the coupled system

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \quad \wedge \quad \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}) \tag{11}$$

is obtained by complete replacement of the signal driver subsystem $\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ in the response system (10).

Instead of completely replacing one of the variables in the system response by its corresponding in transport system, a replacement can be partial as suggested by Guemez and Matthias [15]. In this case, a variable of response system gives rise to its corresponding in transport system only in some of its equations. In general, the stability results in partial replacement differ from those in complete replacement. In this paper it is studied the partial replacement in the nonlinear terms of response system.

# 3 Case study

## 3.1 Using Lorenz systems

**Total diffusive linear bidirectional coupling with partial substitution of $x_2$. (L5 da tese)** Consider the total diffusive linear bidirectional coupling of two identical chaotic Lorenz systems with all the coupling parameters equal to $\rho > 0$ (particular case)

$$\left\{\begin{array}{l} \dot{x}_1 = \sigma(y_1 - x_1) + \rho(x_2 - x_1) \\ \dot{y}_1 = \alpha x_1 - x_1 z_1 - y_1 + \rho(y_2 - y_1) \\ \dot{z}_1 = x_1 y_1 - \beta z_1 + \rho(z_2 - z_1) \end{array}\right. \wedge \left\{\begin{array}{l} \dot{x}_2 = \sigma(y_2 - x_2) + \rho(x_1 - x_2) \\ \dot{y}_2 = \alpha x_2 - \underline{x_1} z_2 - y_2 + \rho(y_1 - y_2) \\ \dot{z}_2 = \underline{x_1} y_2 - \beta z_2 + \rho(z_1 - z_2) \end{array}\right.$$

where it is introduced the partial replacement of variable $x_2$ by the corresponding $x_1$ only in the nonlinear terms $x_2 z_2$ and $x_2 y_2$ of second system. Starting the coupled system from (arbitrary) initial conditions such that $x_1(0) \neq x_2(0)$, $y_1(0) \neq y_2(0)$ and $z_1(0) \neq z_2(0)$, it is reached identical synchronization if the evolution of coupled system evolution is continually confined to a hyperplane $\mathcal{M}$ in phase space. The coordinates $e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$ of synchronization error $\mathbf{e} = (e_x, e_y, e_z)$ in the transversal subspace to $\mathcal{M}$ converge to 0 as $t \to +\infty$ if the point $(0,0,0)$ in the transversal subspace to $\mathcal{M}$ is an asymptotically stable equilibrium point (in this space). This leads to require that the dynamical system in $\mathbf{e}$ defining the transversal perturbations is asymptotically stable at the equilibrium point $(0,0,0)$. Transversal system is defined by the equations

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{y}_2 - \dot{y}_1 \\ \dot{z}_2 - \dot{z}_1 \end{bmatrix} = \begin{bmatrix} \sigma\,(e_y - e_x) - 2\rho e_x \\ \alpha e_x - x_1 e_z - e_y - 2\rho e_y \\ x_1 e_y - \beta e_z - 2\rho e_z \end{bmatrix}.$$

It takes the matricial form $\dot{\mathbf{e}} = \mathbf{A}\,(x_1) \cdot \mathbf{e}$ with

$$\mathbf{A} = \begin{bmatrix} -2\rho - \sigma & \sigma & 0 \\ \alpha & -2\rho - 1 & -x_1 \\ 0 & x_1 & -2\rho - \beta \end{bmatrix}.$$

The main determinants of the matrix

$$\mathbf{A}^T + \mathbf{A} = \begin{bmatrix} -2\,(2\rho + \sigma) & \sigma + \alpha & 0 \\ \sigma + \alpha & -2\,(2\rho + 1) & 0 \\ 0 & 0 & -2\,(2\rho + \beta) \end{bmatrix},$$

are $\Delta_1 = -2\,(2\rho + \sigma)$, $\Delta_2 = 4\,(2\rho + \sigma)\,(2\rho + 1) - (\sigma + \alpha)^2$ and

$$\Delta_3 = -2\left[4\,(2\rho + \sigma)\,(2\rho + 1) - (\sigma + \alpha)^2\right](2\rho + \beta)\,.$$

We have $-\Delta_1 > 0$ and the condition $-\Delta_3 > 0$ is satisfied when/where/if/whenever $\Delta_2 > 0$ (since $2\rho + \beta > 0$). So, we conclude by Proposition 1 that occurs globally stable synchronization if the control and coupling parameters verify the inequality

$$4\,(2\rho + \sigma)\,(2\rho + 1) > (\sigma + \alpha)^2\,.$$

Taking control parameters $\sigma = 10$, $\alpha = 28$ and $\beta = 8/3 = 2.(6)$ this globally stable synchronization condition leads to the threshold of coupling $\rho = 7.1$. We verify that $x_2 \to x_1$, $y_2 \to y_1$ and $z_2 \to z_1$ when systems

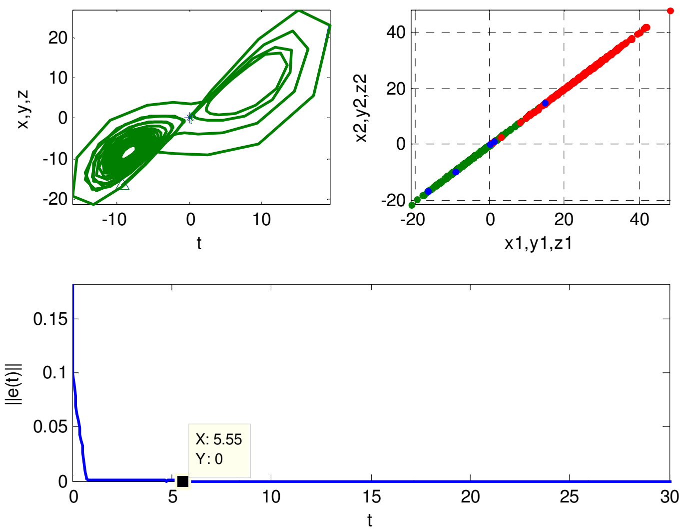


Figure 1: (a) Atractor do sistema ligado; (b) Variedade de sincronizaçao; (c) Evolução do erro de sincronizaçao

evolve (Fig. 1a). After a certain time, the coordinates $x$, $y$ and $z$ of systems verify the equalities $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ (Fig. 1b). So, the distances $|x_2 - x_1|$, $|y_2 - y_1|$ and $|z_2 - z_1|$ converge to 0 over time (Fig. 1c). Equations $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ define a hyperplane $\mathcal{M}$ in the 6-dimensional phase space.Notice that in coupling by negative feedback control with partial replacement of $x_2$ by $x_1$ only in the nonlinear terms $x_2 z_2$ and $x_2 y_2$ of response system **(referir o 1$^o$ artigo, nosso, é o caso L4)**, where the coupling is unidirectional, the threshold of coupling is $\rho = 14.5 > 7.1$.

It is impossible to obtain a stable synchronization condition without partial substitution of $x_2$ by $x_1$ in the nonlinear terms of the second system.

## 3.2 Using Rössler systems

**Total diffusive linear bidirectional coupling. (R3 da tese)** Consider the total diffusive linear bidirectional coupling of two Rössler systems with all the coupling parameters equal to $\rho > 0$ (particular case)

$$\begin{cases} \dot{x}_1 = -(y_1 + z_1) + \rho(x_2 - x_1) \\ \dot{y}_1 = x_1 + ay_1 + \rho(y_2 - y_1) \\ \dot{z}_1 = b + z_1(x_1 - c) + \rho(z_2 - z_1) \end{cases} \wedge \begin{cases} \dot{x}_2 = -(y_2 + z_2) + \rho(x_1 - x_2) \\ \dot{y}_2 = x_2 + ay_2 + \rho(y_1 - y_2) \\ \dot{z}_2 = b + z_2(x_2 - c) + \rho(z_1 - z_2) \end{cases}. \tag{12}$$

Given the synchronization error $\mathbf{e} = (e_x, e_y, e_z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, the transversal system is defined by the equations

$$\begin{cases} \dot{e}_x = \dot{x}_2 - \dot{x}_1 = -e_y - e_z - 2\rho e_x \\ \dot{e}_y = \dot{y}_2 - \dot{y}_1 = e_x + (a - 2\rho)e_y \\ \dot{e}_z = \dot{z}_2 - \dot{z}_1 = z_2 e_x + (x_1 - c - 2\rho)e_z \end{cases}.$$

Consider the Lyapunov function $L(\mathbf{e}) = \left(e_x^2 + e_y^2 + e_z^2\right)/2$ which verifies $L(\mathbf{e}) > 0$ if $\mathbf{e} \neq \mathbf{0}$ and $L(\mathbf{0}) = 0$ for all (values of) $\rho$. It is necessary to determine the strength coupling $\rho$ such that the derivative of $L$ satisfies $\dot{L}(\mathbf{e}) < 0$ if $\mathbf{e} \neq \mathbf{0}$ and $\dot{L}(\mathbf{0}) = 0$. Substituting the expression of $\dot{e}_x$, $\dot{e}_y$ and $\dot{e}_z$ in

$$\dot{L}(\mathbf{e}) = e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z$$

and simplifying, the derivative of $L$ can be written as

$$\begin{aligned} \dot{L}(\mathbf{e}) &= -2\rho e_x^2 + (z_2 - 1)e_x e_z + (a - 2\rho)e_y^2 + (x_1 - c - 2\rho)e_z^2 \\ &\leq -\rho e_x^2 + (a - \rho)e_y^2 + x_1 e_z^2 - (c + \rho)e_z^2 + z_2|e_x e_z| - |e_x e_z|. \end{aligned}$$

Assuming (that) the functions of real variable $x_1$ and $z_2$ are bounded, let $K_x$ and $K_z$ be positive constants such that $|x_1| \leq K_x$ and $|z_2| \leq K_z$. As such it is valid the inequality

$$\dot{L}(\mathbf{e}) \leq -2\rho e_x^2 + (a - 2\rho)e_y^2 + K_x e_z^2 - (c + 2\rho)e_z^2 + K_z|e_x e_z| - |e_x e_z|.$$

(The) Transversal system is asymptotically stable at origin if the constant symmetric matrix

$$\mathbf{P} = \begin{bmatrix} 2\rho & 0 & \frac{1}{2}(1 - K_z) \\ 0 & 2\rho - a & 0 \\ \frac{1}{2}(1 - K_z) & 0 & c + 2\rho - K_x \end{bmatrix},$$

associated with quadratic form $-\|\mathbf{e}\|^T \cdot \mathbf{P} \cdot \|\mathbf{e}\|$, with $\|\mathbf{e}\| = (|e_x|, |e_y|, |e_z|)$, is positive definite. (The) Main determinants $\Delta_i$, $i = 1, 2, 3$, of $\mathbf{P}$ are positive if

$$(2\rho - a) > 0 \quad \wedge \quad (2\rho - a)\left[\rho(c + 2\rho - K_x) - \frac{1}{4}(1 - K_z)^2\right] > 0.$$

As inequality on $\Delta_3$ implies $2\rho - a > 0$ whenever

$$8\rho\,(c + 2\rho - K_x) > (1 - K_z)^2\,,$$

we conclude that the matrix $\mathbf{P}$ is positive definite if

$$2\rho > a \quad \wedge \quad 8\rho\,(2\rho + c - K_x) > (1 - K_z)^2\,.$$

(The) Condition $K_x > 0$ leads to $8\rho\,(2\rho + c) > (1 - K_z)^2$. By the Lyapunov direct method, the synchronization error tends to 0 as $t \to \infty$ whenever the control parameters $a$ and $c$, the coupling strength $\rho$ and the positive constants $K_x$ and $K_z$ limiting the system variables verify the inequalities above, and the systems achieve globally stable synchronization.

Becomes in this way established the following sufficient condition for synchronization.

**Proposition 2** *Two Rössler systems in total diffusive linear bidirectional coupling (12), with a unique parameter coupling $\rho$, achieve globally stable synchronization if*

$$2\rho > a \quad \wedge \quad 8\rho\,(2\rho + c - K_x) > (1 - K_z)^2$$

*where $K_x$ and $K_y$ are positive constants such that $|x_1| \leq K_x$ and $|z_2| \leq K_z$.*

Taking (the control parameters) $a = b = 0.2$ and $c = 5$, we present the Figure 2(a,b,c) obtained for (the coupling strength) $\rho = 8$, which is the lowest value of $\rho$ in a tenth step that verifies the previous inequality established in Proposition 2. As can be seen, the synchronization error evolves rapidly/quickly to 0..

**Total diffusive linear bidirectional coupling with partial replacement. (R4 da tese)** Consider the total diffusive linear bidirectional coupling of two Rössler systems with all the coupling parameters equal to $\rho > 0$ (particular case)

$$\left\{\begin{array}{l} \dot{x}_1 = -\left(y_1 + z_1\right) + \rho\left(x_2 - x_1\right) \\ \dot{y}_1 = x_1 + ay_1 + \rho\left(y_2 - y_1\right) \\ \dot{z}_1 = b + z_1\left(x_1 - c\right) + \rho\left(z_2 - z_1\right) \end{array}\right. \wedge \left\{\begin{array}{l} \dot{x}_2 = -\left(y_2 + z_2\right) + \rho\left(x_1 - x_2\right) \\ \dot{y}_2 = x_2 + ay_2 + \rho\left(y_1 - y_2\right) \\ \dot{z}_2 = b + z_2\left(\underline{x_1} - c\right) + \rho\left(z_1 - z_2\right) \end{array}\right., \tag{13}$$

in which, simultaneously, was introduced the partial replacement of the variable $x_2$ by the corresponding $x_1$ only in the nonlinear term $z_2 x_2$ of second system.

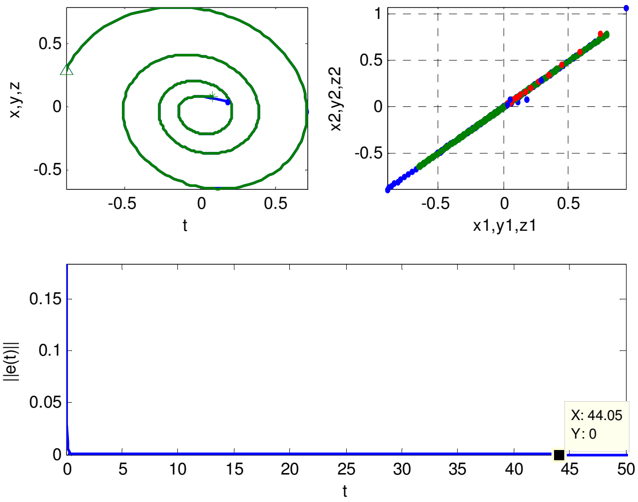


Figure 2: Parameter values $a = b = 0.2$ and $c = 5$; coupling strength $\rho = 8$. (a) Coupled system attractor; (b) Manifold synchronization; (c) Evolution of the synchronization error

Let $e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$ be the components of synchronization error $\mathbf{e}$. For all values of $\rho$ the transversal system is defined by the equations

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{y}_2 - \dot{y}_1 \\ \dot{z}_2 - \dot{z}_1 \end{bmatrix} = \begin{bmatrix} -e_y - e_z - 2\rho e_x \\ e_x + (a - 2\rho)\, e_y \\ (x_1 - c - 2\rho)\, e_z \end{bmatrix}.$$

Consider the Lyapunov function $L(\mathbf{e}) = \left(e_x^2 + e_y^2 + e_z^2\right)/2$ which verifies $L(\mathbf{e}) > 0$ if $\mathbf{e} \neq \mathbf{0}$ and $L(\mathbf{0}) = 0$ for all values of $\rho$. It is necessary to determine the strength coupling $\rho$ such that the derivative of $L$ satisfies $\dot{L}(\mathbf{e}) < 0$ if $\mathbf{e} \neq \mathbf{0}$ and $\dot{L}(\mathbf{0}) = 0$. Substituting the expression of $\dot{e}_x$, $\dot{e}_y$ and $\dot{e}_z$ in

$$\dot{L}(\mathbf{e}) = e_x \dot{e}_x + e_y \dot{e}_y + e_z \dot{e}_z$$

and simplifying, the derivative of $L$ can be written as

$$\begin{aligned} \dot{L}(\mathbf{e}) &= -2\rho e_x^2 - e_x e_z + (a - 2\rho)\, e_y^2 + (x_1 - c - 2\rho)\, e_z^2 \\ &\leq -2\rho e_x^2 + (a - 2\rho)\, e_y^2 + x_1 e_z^2 - (c + 2\rho)\, e_z^2 - |e_x e_z|\,. \end{aligned}$$

Assuming (that) the functions of real variable $x_1$ is bounded, let $K_x$ be positive constant such that $|x_1| \leq K_x$. As such it is valid the inequality

$$\dot{L}(\mathbf{e}) \leq -2\rho e_x^2 + (a - 2\rho)\, e_y^2 + K_x e_z^2 - (c + 2\rho)\, e_z^2 - |e_x e_z|\,.$$

For the transversal system to be asymptotically stable at origin, the constant symmetric matrix

$$\mathbf{P} = \begin{bmatrix} 2\rho & 0 & 0.5 \\ 0 & 2\rho - a & 0 \\ 0.5 & 0 & c + 2\rho - K_x \end{bmatrix}$$

(associated with quadratic form $-\|\mathbf{e}\|^T \cdot \mathbf{P} \cdot \|\mathbf{e}\|$, with $\|\mathbf{e}\| = (|e_x|, |e_y|, |e_z|)$,) must be positive definite. The main determinants $\Delta_i$, $i = 1, 2, 3$, of $\mathbf{P}$ are positive if

$$2\rho - a > 0 \quad \wedge \quad (2\rho - a)\left[2\rho\,(c + 2\rho - K_x) - \frac{1}{4}\right] > 0.$$

As inequality on $\Delta_3$ implies $2\rho - a > 0$ whenever

$$8\rho\,(c + 2\rho - K_x) > 1,$$

we conclude that the matrix $\mathbf{P}$ is positive definite if

$$2\rho > a \quad \wedge \quad 8\rho\,(2\rho + c - K_x) > 1.$$

(The) Condition $K_x > 0$ leads to $8\rho\,(2\rho + c) > 1$. By the Lyapunov direct method, $\mathbf{e} \to 0$ as $t \to \infty$ whenever the control parameters $a$ and $c$, the coupling strength $\rho$ and the positive constant $K_x$ limiting the system variable verify the inequalities above, and the systems achieve globally stable synchronization.

So, it is valid the following result.

**Proposition 3** *Two Rössler systems in the coupling (13) achieve globally stable synchronization if*

$$2\rho > a \quad \wedge \quad 8\rho\,(2\rho + c - K_x) > 1$$

*where $K_x$ is a positive constant such that $|x_1| \leq K_x$.*

(The) Figure 3(a,b,c), whose graphs were obtained with $\rho = 6$, which is the lowest value of $\rho$ in a tenth step that verifies the previous inequality,

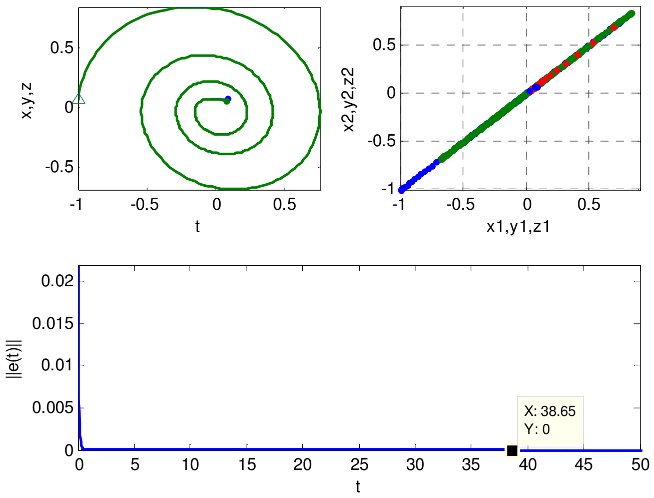


Figure 3: Parameter values $a = b = 0.2$ and $c = 5$; coupling strength $\rho = 8$. (a) Coupled system attractor; (b) Manifold synchronization; (c) Evolution of the synchronization error

shows what is established in Proposition 3. We observe a rapid evolution of synchronization error to 0.

**Partial diffusive linear bidirectional coupling. (R5 da tese) (vale a pena pôr ou só referir na conclusão?)**Consider the partial linear diffusive linear bidirectional coupling of two Rössler systems

$$\left\{ \begin{array}{l} \dot{x}_1 = -(y_1 + z_1) + \rho (x_2 - x_1) \\ \dot{y}_1 = x_1 + a y_1 \\ \dot{z}_1 = b + z_1 (x_1 - c) \end{array} \right. \wedge \left\{ \begin{array}{l} \dot{x}_2 = -(y_2 + z_2) + \rho (x_1 - x_2) \\ \dot{y}_2 = x_2 + a y_2 \\ \dot{z}_2 = b + z_2 (x_2 - c) \end{array} \right. . \tag{14}$$

Let $e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$ be the components of synchronization error $\mathbf{e}$. For all values of $\rho$ the transversal system is defined by the equations

$$\left\{ \begin{array}{l} \dot{e}_x = \dot{x}_2 - \dot{x}_1 = -e_y - e_z - 2\rho e_x \\ \dot{e}_y = \dot{y}_2 - \dot{y}_1 = e_x + a e_y \\ \dot{e}_z = \dot{z}_2 - \dot{z}_1 = z_2 e_x + (x_1 - c) e_z \end{array} \right. .$$

Consider the Lyapunov function

$$L(\mathbf{e}) = \frac{1}{2}\left(e_x^2 + e_y^2 + e_z^2\right)$$

which verifies $L(\mathbf{e}) > 0$ if $\mathbf{e} \neq \mathbf{0}$ and $L(\mathbf{0}) = 0$ for all values of $\rho$. It is necessary to determine the strength coupling $\rho$ such that the derivative of $L$ satisfies $\dot{L}(\mathbf{e}) < 0$ if $\mathbf{e} \neq \mathbf{0}$ and $\dot{L}(\mathbf{0}) = 0$. Substituting the expression of $\dot{e}_x$, $\dot{e}_y$ and $\dot{e}_z$ in

$$\dot{L}(\mathbf{e}) = e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z$$

and simplifying, the derivative of $L$ can be written as

$$\begin{array}{rcl} \dot{L}(\mathbf{e}) & = & -2\rho e_x^2 + (z_2 - 1)\, e_x e_z + a e_y^2 + (x_1 - c)\, e_z^2 \\ & \leq & -2\rho e_x^2 + a e_y^2 + x_1 e_z^2 - c e_z^2 + z_2 \left|e_x e_z\right| - \left|e_x e_z\right|. \end{array}$$

Assuming (that) the functions of real variable $x_1$ and $z_2$ are bounded, let $K_x$ and $K_z$ be positive constants such that $|x_1| \leq K_x$ and $|z_2| \leq K_z$. As such it is valid the inequality

$$\dot{L}(\mathbf{e}) \leq -2\rho e_x^2 + a e_y^2 + K_x e_z^2 - c e_z^2 + K_z \left|e_x e_z\right| - \left|e_x e_z\right|.$$

(The) Transversal system is asymptotically stable at origin if the constant symmetric matrix

$$\mathbf{P} = \begin{bmatrix} 2\rho & 0 & \frac{1}{2}(1 - K_z) \\ 0 & -a & 0 \\ \frac{1}{2}(1 - K_z) & 0 & c - K_x \end{bmatrix}$$

associated with quadratic form $-\left\|\mathbf{e}\right\|^T \cdot \mathbf{P} \cdot \left\|\mathbf{e}\right\|$, with $\left\|\mathbf{e}\right\| = (|e_x|, |e_y|, |e_z|)$, is positive definite. (The) Main determinants $\Delta_i$, $i = 1, 2, 3$, of $\mathbf{P}$ are positive if

$$-2a\rho > 0 \quad \wedge \quad -8a\rho\,(c - K_x) + a\,(1 - K_z)^2 > 0.$$

Since the condition concerning the $\Delta_2$ is impossible, the Lyapunov direct method is not conclusive with this choice of Lyapunov function.

# 4 Conclusions

The total diffusive linear bidirectional coupling between chaotic identical Lorenz systems and between chaotic identical Rössler systems allowed to obtain sufficient conditions to globally stable synchronization. In some cases it is considered simultaneously the partial replacement of the variable $x_1$ by their corresponding $x_2$, only in the nonlinear terms of second system. Such partial replacement only reveal advantage.

In coupling between Lorenz systems the globally stable synchronization condition result from the classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite (Proposition 1), where $\mathbf{A}$ is the matrix characterizing the transversal system of coupling. It was not possible to apply this proposition in coupling betwwen Rössler systems by performing or not the partial replacement on the nonlinear term of second system.

In couplings between Rössler systems these conditions are based on (the) increase/accretion of derivative of an appropriate Lyapunov function assuming yet the limitation of certain variables. When the partial replacement is carried out such limitation is necessary on only one variable ($x_1$), while in the absence of replacing two variables are considered ($x_1$ and $z_2$). Also the partial replacement allows to obtain a lower threshold of coupling. Noting that if the diffusive linear coupling is partial then it is not guaranteed by this approach any sufficient condition to globally stable synchronization.

# References


[1] T. Yamada and H. Fujisaka (1983), Stability theory of synchronized motion in coupled-oscillator systems, *Prog. Theoret. Phys.* **69** (1), 32-47.

[2] A.S. Pikovsky (1984), On the interaction of strange attractors, *Z. Physik B* **55** (2), 149-154.

[3] L.M. Pecora and T.L. Carroll (1990), Synchronization in chaotic systems, *Phys. Rev. Lett.* **64** (8), 821-824.

[4] L.M. Pecora, T.L. Carroll, G.A.. Johnson, D.J. Mar e J.F. Heagy (1997), Fundamentals of synchronization in chaotic systems, concepts, and applications, *Chaos* **7** (4), 520-543.

[5] T. Yamada e H. Fujisaka (1983), Stability theory of synchronized motion in coupled-oscillator systems. II, *Prog. Theoret. Phys.* **70** (5), 1240-1248.

[6] R. He and P.G. Vaidya (1992), Analysis and synthesis of synchronous periodic and chaotic systems, *Phys. Rev. A* **46** (12), 7387-7392.

[7] A. Stefański (2000), Estimation of the largest Lyapunov exponent in systems with impacts, *Chaos, Solitons and Fractals* **11,** 2443-2451.

[8] A. Stefański e T. Kapitaniak (2003), Synchronization of two chaotic oscillators via a negative feedback mechanism, *Int. J. Solids Structures* **40,** 5175-5185.

[9] J.C. Alexander, I. Kan, J.A. Yorke e Z. You (1992), Riddled basins, *Int. J. Bifur. Chaos* **2** 795-813.

[10] J.C. Sommerer e E. Ott (1993), A physical system with qualitatively uncertain dynamics, *Nature* **365,** 136-140.

[11] E. Ott, J.C. Sommerer, J.C. Alexander, I. Kan e J.A. Yorke (1994), The transition to chaotic attractors with riddled basins, *Physica D* **76,** 384-410.

[12] T. Kapitaniak (1994), Synchronization of chaos using continuous control, *Phys. Rev. E* **50,** 1642-1644.

[13] P. Ashwin, J. Buescu e I. Stewart (1994), Bubbling of attractors and synchronization of oscillators*, Phys. Lett. A* **193,** 126-139.

[14] H.E. Nusse e J.A. Yorke (1991), A numerical procedure for finding accessible trajectories on basin boundaries, *Nonlinearity* **4,** 1183-1212.

[15] J. Guemez e M.A. Matias (1995), Modified method for synchronizing and cascading chaotic systems, *Phys. Rev. E* **52,** R2145-R2148.