The Hyperbolic Sieve of Primes and Products xy

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October 9, 2023

The Hyperbolic Sieve of Prime Numbers.

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First text: 12 June 2020 **version:** 8 October 2023 19:49:38

Abstract.

We start this study producing the **HL - Hyperbolic Lattice Grid** in the form of $HL[x, y] = x * y$. Then we show that the **SMT – Square Multiplication Table** is the result of the integer coordinates of the **HL - Hyperbolic Lattice Grid** in the form of $HL[x, y] = x * y$, in the first quadrant. From the SMT we define the **SMTSP – Square Multiplication Table Sieve of Primes**. Then we show the SMT covered by the quadratic sequences in the form of $x = y(y \pm b)$. Then, we expand the SMT to the **FMT - Full Multiplication Table**. Because of the FMT, we define all integers in terms of the **Pairs of Complementary Divisors** $(x; y)$. We make a disambiguation between factors and divisors. From these properties, we created the **TMTSP - Triangular Multiplication Table Sieve of Prime Numbers**. The **Pairs of Complementary Divisors** $(x; y)$ of integers are exactly the pairs of integer coordinates (x, y) of the Cartesian points of TMTSP in the XY-plane. From the TMT we define the **TMTSP – Triangular Multiplication Table Sieve of Primes**. The multiplication table proves that all integers must be classified as primes or composites, no exception. There is an equivalence (or isomorphism) between the **Hyperbolic Sieve of Primes** with the **Parabolic Sieve of Primes.** Both produce identical results.

Keywords.

Sieve of prime numbers; primes and composites; multiplication table; pair of complementary divisors; hyperbolic lattice grid.

2010 Mathematics Subject Classification.

11N32; 11N35; 11N36; 11A05; 35L02.

1. Introduction.

Just as there is the "Parabolic Sieve of Primes" discovered by Matiyasevich, Yuri and Stechkin, Boris. (1999) "A visual Sieve for prime Numbers", available online at https://logic.pdmi.ras.ru/~yumat/personaljournal/sieve/sieve.gif, here we are going to develop the "Hyperbolic Sieve of Primes". Everything is based on the multiplication table.

2. The HL - Hyperbolic Lattice Grid.

The two steps to construct the HL - Hyperbolic Lattice Grid in the XY-plane:

1. Draw all the circles as $x^2 + y^2 = n$ where $n \geq 0$ in integer. Each circle has a radius $r = \sqrt{n}$.
2. For each $r = \sqrt{n}$, all the vertical lines are $x = r = \sqrt{n}$, and all the horizontal lines are $y = r = \sqrt{n}$.

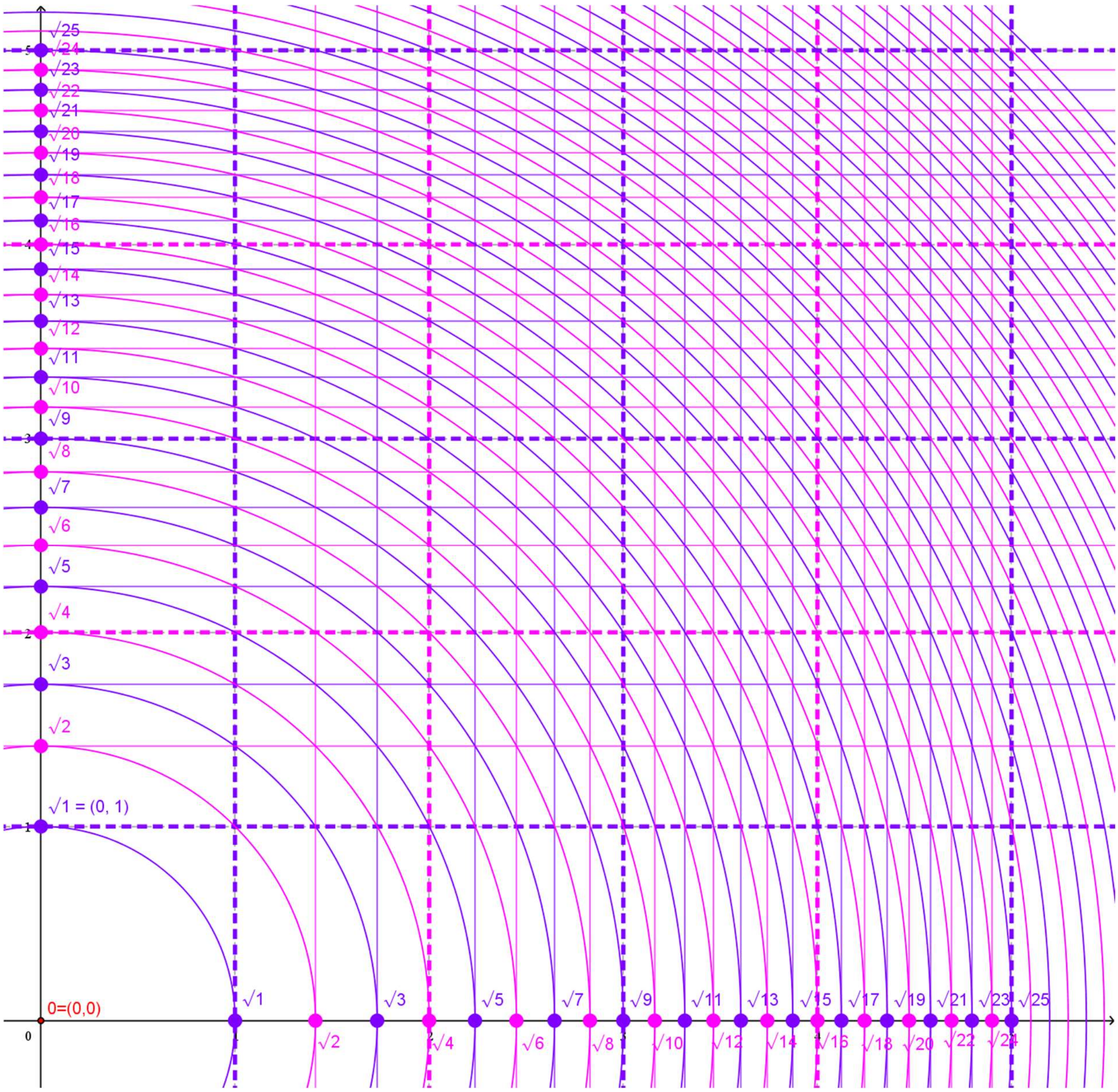


Figure 1. C000443 The HL - Hyperbolic Lattice Grid construction in the 1st quadrant of the XY-plane.

For $n = even$, we use the magenta web color #FF00FF.

For $n = odd$, we use the violet web color #7F00FF.

Each intersection of the vertical lines, the horizontal lines, and the circles generates a point. For each point, we assign the value $HL[x, y]$ which is the product of its X and Y-coordinates.

Any point in the XY-plane has the value defined by the function $HL[x, y] = x * y$.

See below the points with integer (x, y) coordinates, and the points with the coordinates $x = y$ over the diagonal line 45º.

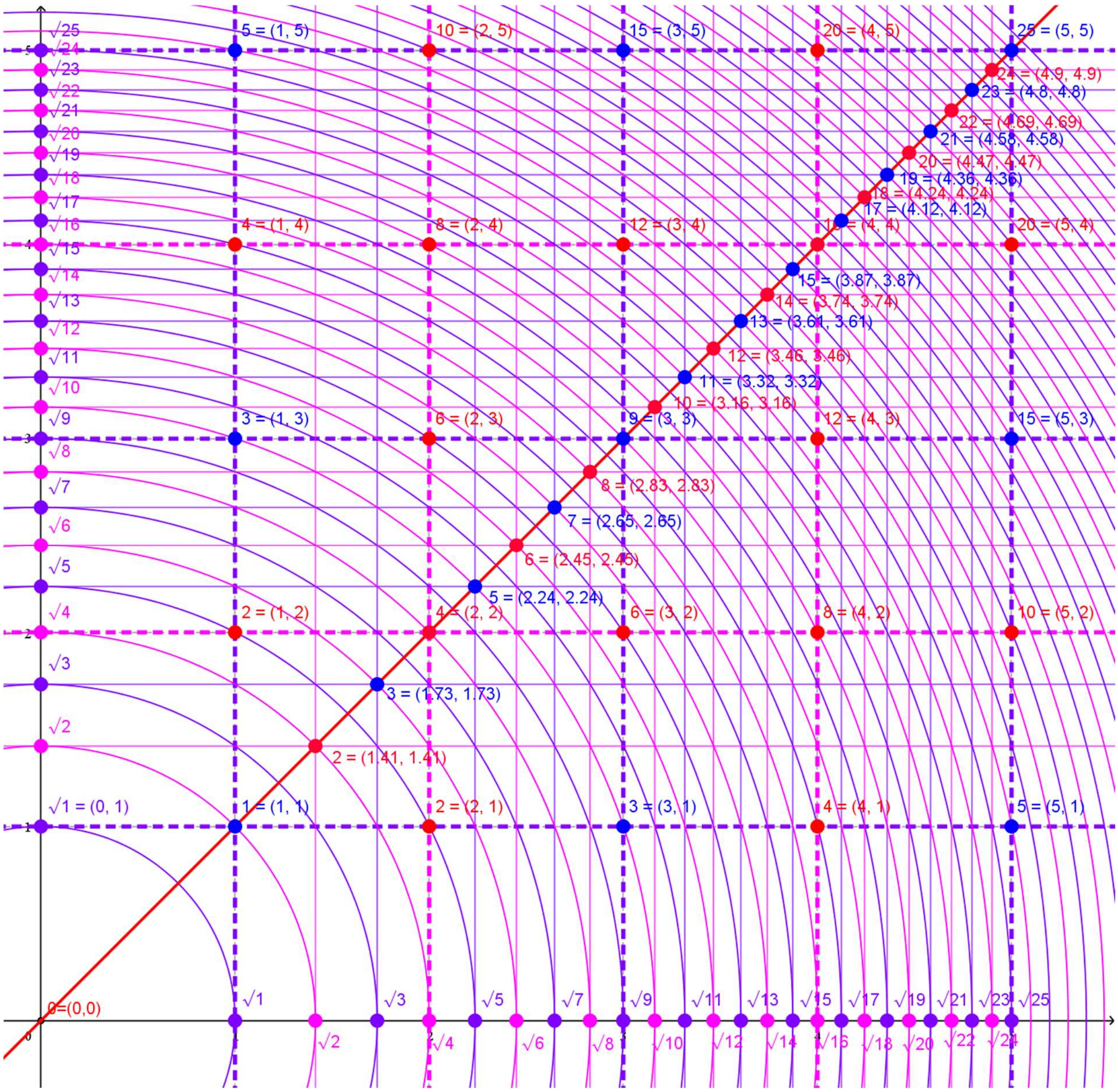


Figure 2. C000443 The integers point products: the even points in red, the odd points in blue.

The 45º diagonal in red is the diagonal of the square numbers because it is the only one that has $x = y$. Thus, the product $x * y = x^2 = y^2$ is always a square. The integer coordinate points (x, y) produce the sequence https://oeis.org/A000290.

Because all the plane is a hyperbolic lattice grid, we can draw the hyperbolas that are the result of the products $HL[x, y] = xy = Integer$.

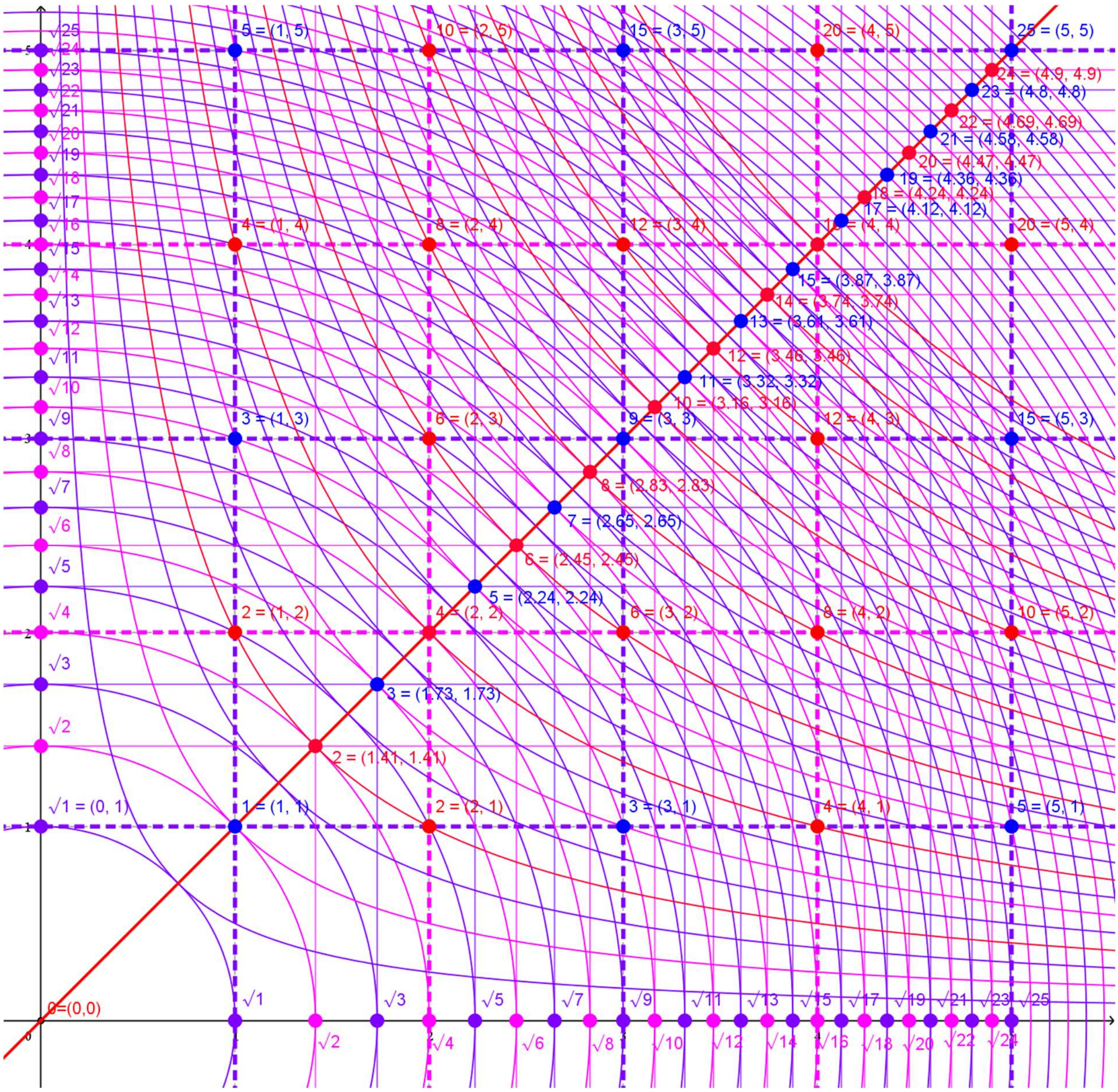


Figure 3. C000443 The HL - hyperbolic lattice grid with its hyperbolas as $HL[x, y] = xy = \frac{integer}{2}$ in the 1st quadrant.

The 45° diagonal line $x = y$ is the transverse axis of all the hyperbolas in the form of $HL[x, y] = xy$.

The violet web color #7F00FF lines represent the hyperbolas $HL[x, y] = xy = \frac{odd}{2} = integer \pm 0.5$. Hyperbolas $HL[x, y] = xy = \frac{odd}{2} = integer \pm 0.5$ are tangent to the circles with radius $r^2 = (odd\ number)^2$. These hyperbolas cross points with one of the two coordinates equal to $(integer \pm 0.5)$.

The magenta web color #FF00FF and red lines represent the hyperbolas $HL[x,y] = xy = \frac{even}{2} = integer$. Hyperbolas $HL[x,y] = xy = \frac{even}{2} = integer$ are tangent to the circles with radius $r^2 = (even\ number)^2$. Only these hyperbolas cross the Cartesian points with both XY integer coordinates.

3. The SMT - Square Multiplication Table.

See the hyperbolas intersecting all the Cartesian points of the XY plane with integer coordinates. See the result in the figure below where we keep the diagonal 45º just for reference.

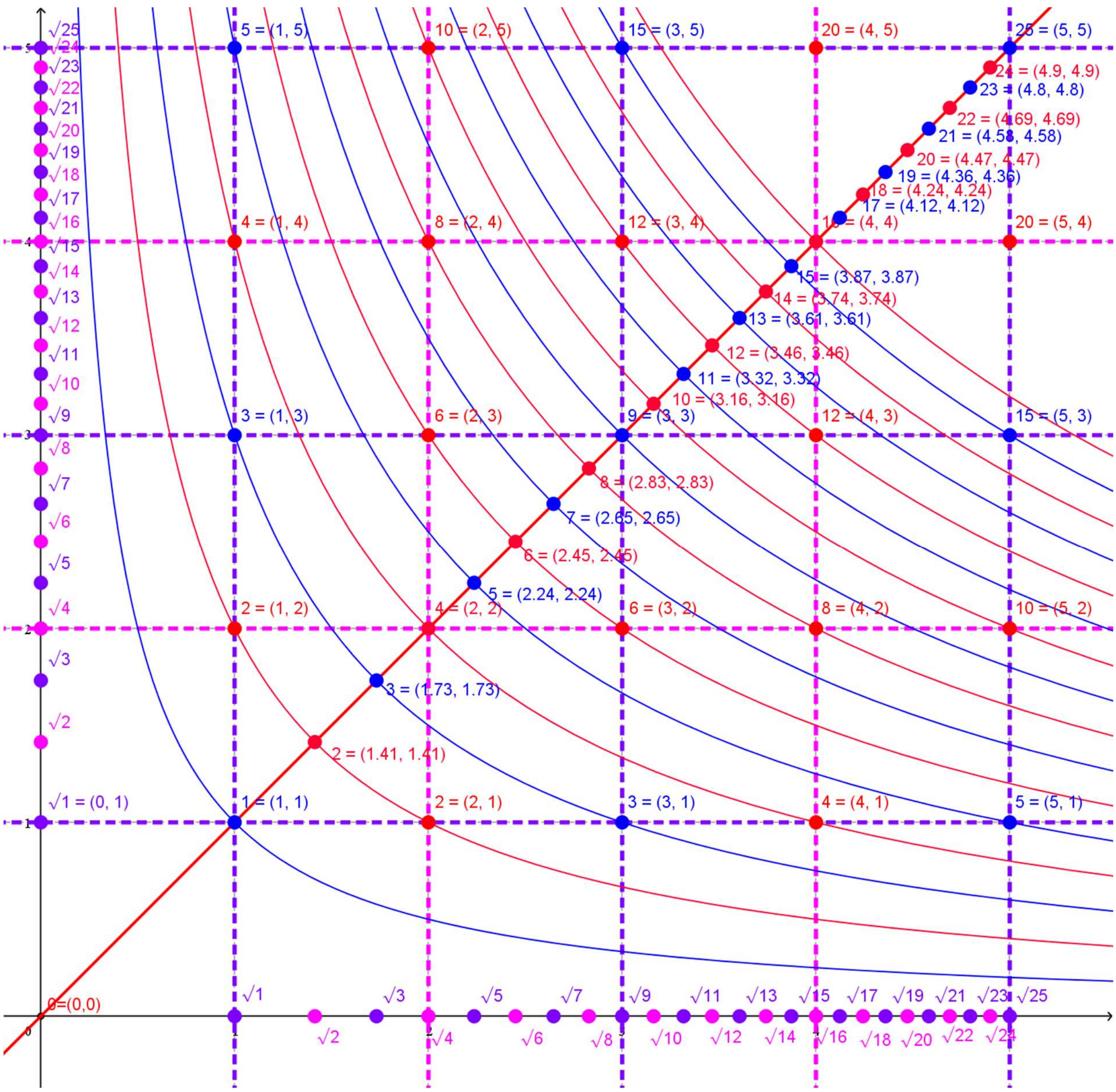


Figure 4. C000443 The HL - hyperbolic lattice grid with the hyperbolas crossing all points with integer coordinates $HL[x,y] = xy = integer$ in the 1st quadrant.

What we're looking at is exactly the multiplication table in the 1st quadrant of the XY plane. This is the hyperbolic lattice grid of the SMT - Square Multiplication Table.

The first use of the name SMT – Square Multiplication Table in OEIS is dated June 25, 2001, at sequences https://oeis.org/A062856 and https://oeis.org/A062857. Mentioned only as a multiplication table, the sequence https://oeis.org/A003991 is registered in the OEIS history on March 15, 1996.

Some related links: https://oeis.org/A004247 and https://oeis.org/A061017.

For example, the integer 4 has three points with integer coordinates in this 1st quadrant:

1. $(x,y) = (\sqrt{1}, \sqrt{16}) = (1,4)$, and $HL[x,y] = 1 * 4 = 4$.
2. $(x,y) = (\sqrt{4}, \sqrt{4}) = (2,2)$, and $HL[x,y] = 2 * 2 = 4$.
3. $(x,y) = (\sqrt{16}, \sqrt{1}) = (4,1)$, and $HL[x,y] = 4 * 1 = 4$.

The transverse axis 45° diagonal line $x = y$ is the symmetry line of the HL - hyperbolic lattice grid in the 1st and 3rd quadrant of the XY-plane.

3.1. C000XXX The SMT in table format.

See the SMT in table format:

3.2. C001741 Bijection between SMT and A061017.

Because https://oeis.org/A027750 * https://oeis.org/A056538 = https://oeis.org/A061017, then the bijection between the products in SMT and https://oeis.org/A061017 comes from the figure:

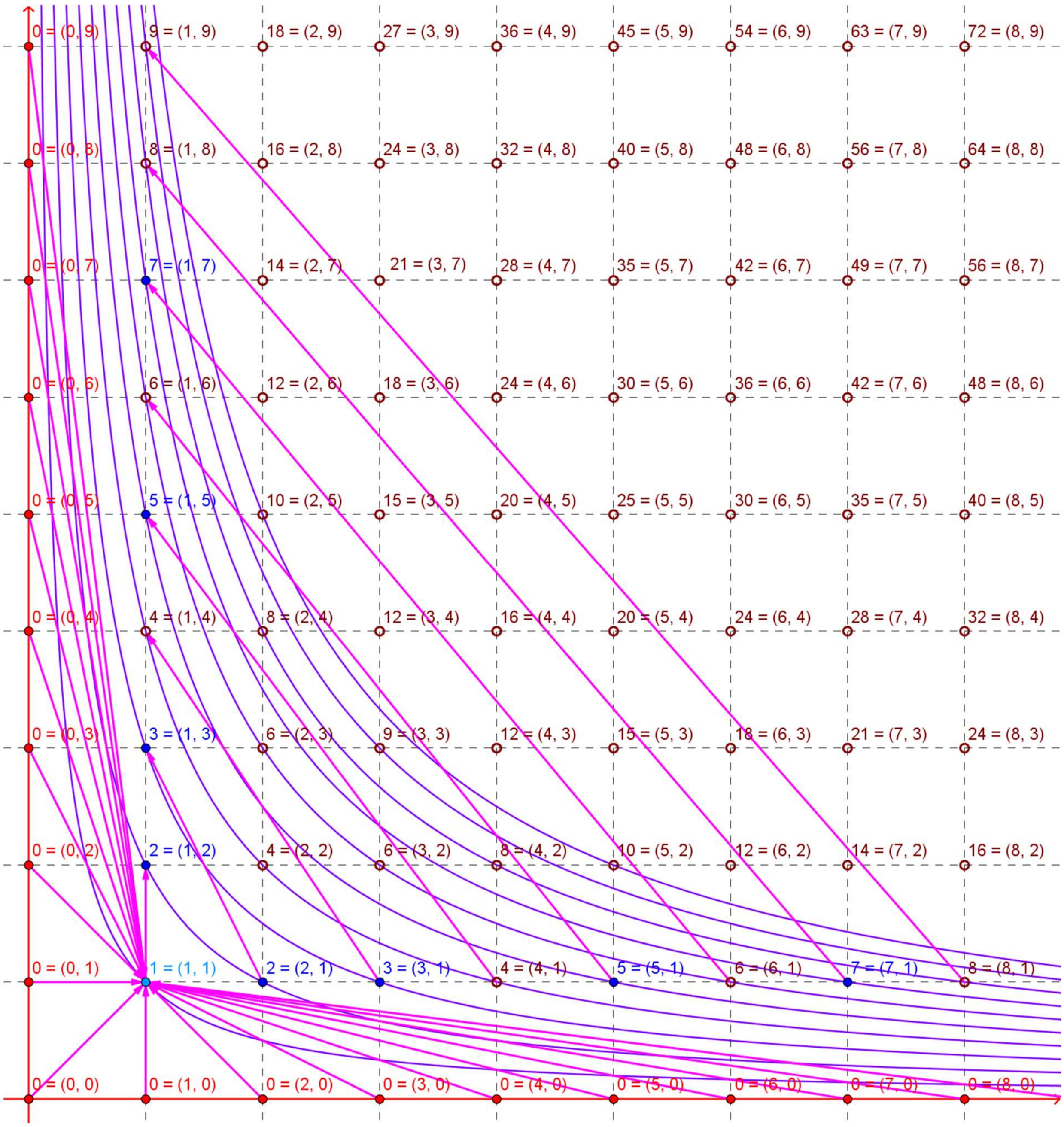


Figure 5. C001741 Bijection between https://oeis.org/A061017 and the SMT – Square Multiplication Table.

From any point 0 in X-axis or Y-axis, the pink vectors lead us to the point 1. The https://oeis.org/A061017 is the result of path formed by the pink vectors that begin in the point 1. Points that are not in the integer Cartesian coordinates are disregarded.

Pink vectors from bottom to top always points from element $(x * 1)$ to $\left(1 * (x + 1)\right)$. They start in row $y = 1$ and end at the column $x = 1$.

The hyperbola trail starts at column $x = 1$ and ends at row $y = 1$. Hyperbolas connect all the elements of the XY-plane that have value $x * y$ integer.

3.3. C000441 The number of products in SMT.

Because https://oeis.org/A027750 * https://oeis.org/A056538 = https://oeis.org/A061017, then the number of products in SMT comes from the table:

	x coordinate		y coordinate		products	number of products
Tally	A027750	*	A056538	=	A061017	A000005
1	1	*	1	=	1	1
2	1	*	2	=	2	2
3	2	*	1	=	2	
4	1	*	3	=	3	2
5	3	*	1	=	3	
6	1	*	4	=	4	3
7	2	*	2	=	4	
8	4	*	1	=	4	
9	1	*	5	=	5	2
10	5	*	1	=	5	
11	1	*	6	=	6	4
12	2	*	3	=	6	
13	3	*	2	=	6	
14	6	*	1	=	6	
15	1	*	7	=	7	2
16	7	*	1	=	7	
17	1	*	8	=	8	4
18	2	*	4	=	8	
19	4	*	2	=	8	
20	8	*	1	=	8	
21	1	*	9	=	9	3
22	3	*	3	=	9	
23	9	*	1	=	9	
24	1	*	10	=	10	4
25	2	*	5	=	10	
26	5	*	2	=	10	
27	10	*	1	=	10	

Figure 6. C000441 The number of products in SMT.

The last column is the number of products in SMT. It is the sequence https://oeis.org/A000005.

3.4. The SMTSP – Square Multiplication Table Sieve of Primes.

Each positive integer number has a unique representation in the 45° diagonal line transverse axis $x = y$ in the 1st and 3rd quadrant.

The prime numbers are only on the $x = 1$ and $y = 1$ lines. They have the hyperbolas passing through only points with integer coordinates in the form of $(1, Prime)$ or $(Prime, 1)$.

Only composite numbers have hyperbolas crossing more than 2 points with integer coordinates.

Only square numbers have integer coordinates along the diagonal of 45°. Because of that only square numbers have an odd number of divisors. All other integers that are not square numbers have an even number of divisors. The number 0 is a square number but is undefined if it has an even or odd number of divisors.

Only the square of prime numbers has hyperbolas crossing 3 points with integer coordinates.

Only the composite numbers different from the square of primes have hyperbolas crossing more than 3 points with integer coordinates.

- The point $(x, y) = (\sqrt{0}, \sqrt{0}) = (0,0)$ in the diagonal line transverse axis $x = y$ is the integer 0. Because the hyperbola $HL[x, y] = xy = 0$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and intersect with infinitely many other points with integer XY-coordinates, then number 0 is a composite number.
- The point $(x, y) = (\sqrt{1}, \sqrt{1}) = (1,1)$ in the diagonal line transverse axis $x = y$ is the integer 1. Because the hyperbola $HL[x, y] = xy = 1$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and does not intersect with any other point with integer XY-coordinates, then number 1 is a prime number.
- The point $(x, y) = (\sqrt{2}, \sqrt{2})$ in the diagonal line transverse axis $x = y$ is the integer 2. Because the hyperbola $HL[x, y] = xy = 2$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and does not intersect with any other point with integer XY-coordinates, then number 2 is a prime number.
- The point $(x, y) = (\sqrt{3}, \sqrt{3})$ in the diagonal line transverse axis $x = y$ is the integer 3. Because the hyperbola $HL[x, y] = xy = 3$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and does not intersect with any other point with integer XY-coordinates, then number 3 is a prime number.
- The point $(x, y) = (\sqrt{4}, \sqrt{4}) = (2,2)$ in the diagonal line transverse axis $x = y$ is the integer 4. Because the hyperbola $HL[x, y] = xy = 4$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and intersect with other point with integer XY-coordinates, then number 4 is a composite number.
- The point $(x, y) = (\sqrt{5}, \sqrt{5})$ in the diagonal line transverse axis $x = y$ is the integer 5. Because the hyperbola $HL[x, y] = xy = 5$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and does not intersect with any other point with integer XY-coordinates, then number 5 is a prime number.
- The point $(x, y) = (\sqrt{6}, \sqrt{6})$ in the diagonal line transverse axis $x = y$ is the integer 6. Because the hyperbola $HL[x, y] = xy = 6$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and intersect with other points with integer XY-coordinates, then number 6 is a composite number.
- The point $(x, y) = (\sqrt{7}, \sqrt{7})$ in the diagonal line transverse axis $x = y$ is the integer 7. Because the hyperbola $HL[x, y] = xy = 7$ crosses the lines $x = 1$ and $y = 1$ in integer

coordinates and does not intersect with any other point with integer XY-coordinates, then number 7 is a prime number.

- The point $(x,y) = (\sqrt{8},\sqrt{8})$ in the diagonal line transverse axis $x = y$ is the integer 8. Because the hyperbola $HL[x,y] = xy = 8$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and intersect with other points with integer XY-coordinates, then number 8 is a composite number.
- The point $(x,y) = (\sqrt{9},\sqrt{9}) = (3,3)$ in the diagonal line transverse axis $x = y$ is the integer 9. Because the hyperbola $HL[x,y] = xy = 9$ crosses the lines $x = 1$ and $y = 1$ in integer coordinates and intersect with other point with integer XY-coordinates, then number 9 is a composite number.
- And so on…

3.5. C001169 The SMT in the table format.

Here's what the multiplication table looks like in table format.

20	0	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400
19	0	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
18	0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
17	0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
16	0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
15	0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
14	0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
13	0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
12	0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
11	0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
10	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
9	0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
8	0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
7	0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
6	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
4	0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
3	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Figure 7. C001169 The SMT in table format.

From the 45º diagonal line of the square numbers, we have the parallel diagonal line of the oblong numbers, then the diagonal line of the (square minus 1) numbers, and so on.

We show the area that is free of repetition from the diagonal of square numbers.

3.6. The quadratic sequences in the form of $x = y(y \pm b)$.

From the diagonal of squares, diagonal of oblongs and diagonal (square minus one) numbers, notice how evident it is the diagonal lines ±45º with all quadratic sequences in the form of $x = y(y \pm b)$.

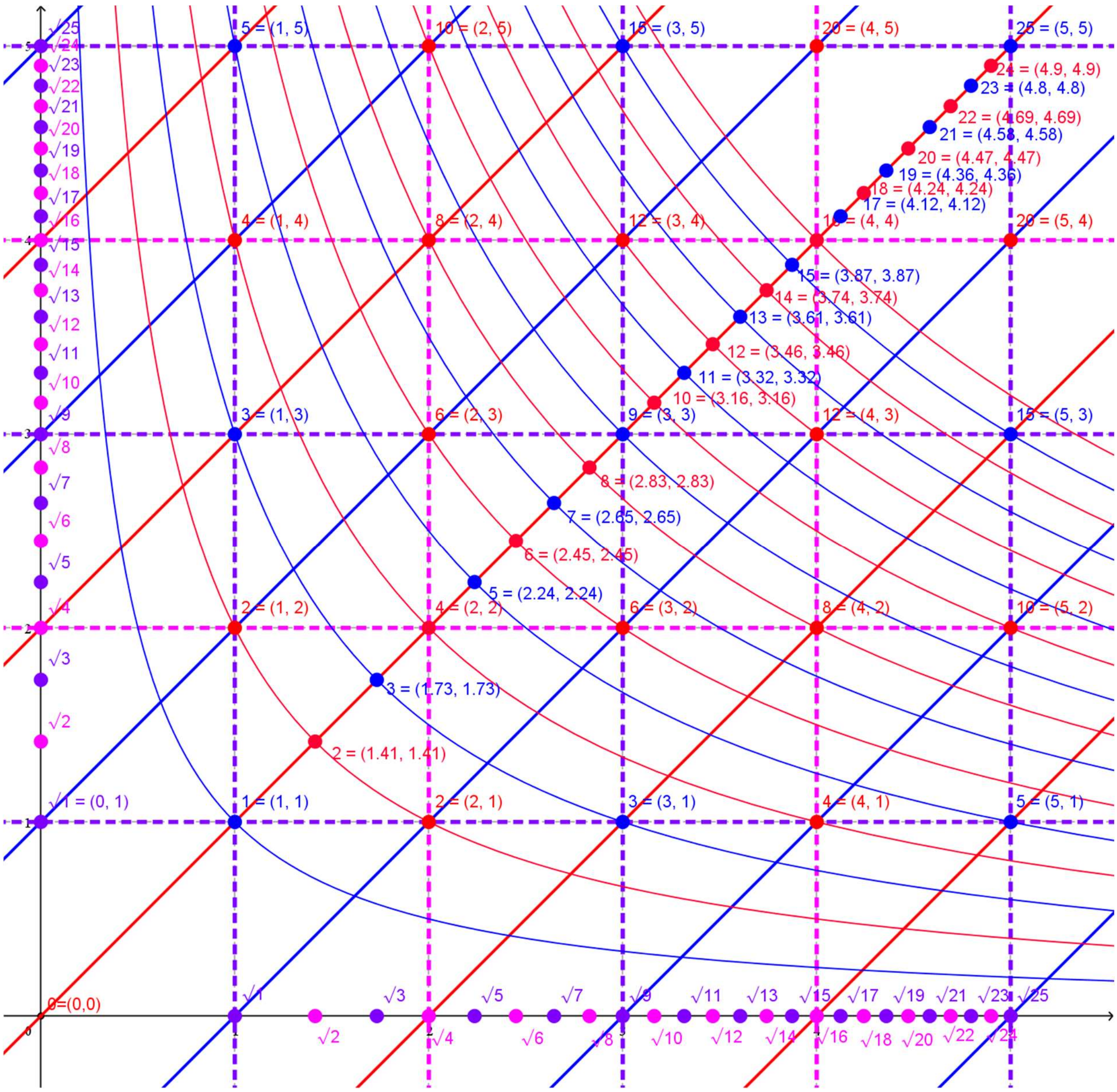


Figure 8. C001169 The quadratic lines in SMT.

The red diagonals are in the form of (square1 minus square2) numbers.

The blue diagonals are in the form of (oblong1 minus oblong2) numbers.

Only the red diagonals have odd numbers.

Another way to see the unique property of prime and composite numbers is to understand their appearance in the equations in the form of $n = x(x + y)$. See post "Understanding A056737 and the quadratic sequences: not so "quibbles"" available online at https://www.mersenneforum.org/showpost.php?p=597461&postcount=14 .

To make the reasoning easier, let us assume x and y non-negative.

The number 1 is the "oddest prime number" or the "first positive prime number" because it appears in the sequence $n = x(x + 0)$ of the square numbers https://oeis.org/A000290, and does not appear in any other sequence in the form of $n = x(x + y)$.

The number 2 is the "second positive prime number" because it appears in the sequence $n = x(x + 1)$ of the oblong numbers https://oeis.org/A002378, and does not appear in any other previous sequence in the form of $n = x(x + y)$ for $0 \leq y < 1$.

The number 3 is the "third positive prime number" because it appears in the sequence $n = x(x + 2)$ (square minus 1) numbers https://oeis.org/A005563, and it does not appear in any other previous sequence in the form of $n = x(x + y)$ for $0 \leq y < 2$.

The number 4 is a composite because it appears in the sequence $n = x(x + 3)$ (oblong minus 2) numbers https://oeis.org/A028552, and also in the previous sequence $n = x(x + 0)$ of squares https://oeis.org/A000290.

The number 5 is a prime number because it appears in the sequence $n = x(x + 4)$ (square minus 4) numbers https://oeis.org/A028347, and does not appear in any other previous sequence in the form of $n = x(x + y)$ for $0 \leq y < 4$.

The number 6 is a composite because it appears in the sequence $n = x(x + 5)$ (oblong minus 6) numbers https://oeis.org/A028557, and also in the previous sequence $n = x(x + 1)$ of the oblongs https://oeis.org/A002378.

The number 7 is a prime number because it appears in the sequence $n = x(x + 6)$ (square minus 9) numbers https://oeis.org/A028560, and it does not appear in any other previous sequence in the form of $n = x(x + y)$ for $0 \leq y < 6$.

...and so on.

Finally, the number 0 appears in all the above sequences. It is "the oddest composite number".

(We can take out the word "previous" from "previous sequences", but let's keep it. That's because if it appeared subsequently, it would be a multiple of the prime which would be a composite.)

4. The FMT – Full Multiplication Table.

Now, let's expand SMT to the entire XY plane. So, we get what we call FMT – Full Multiplication Table.

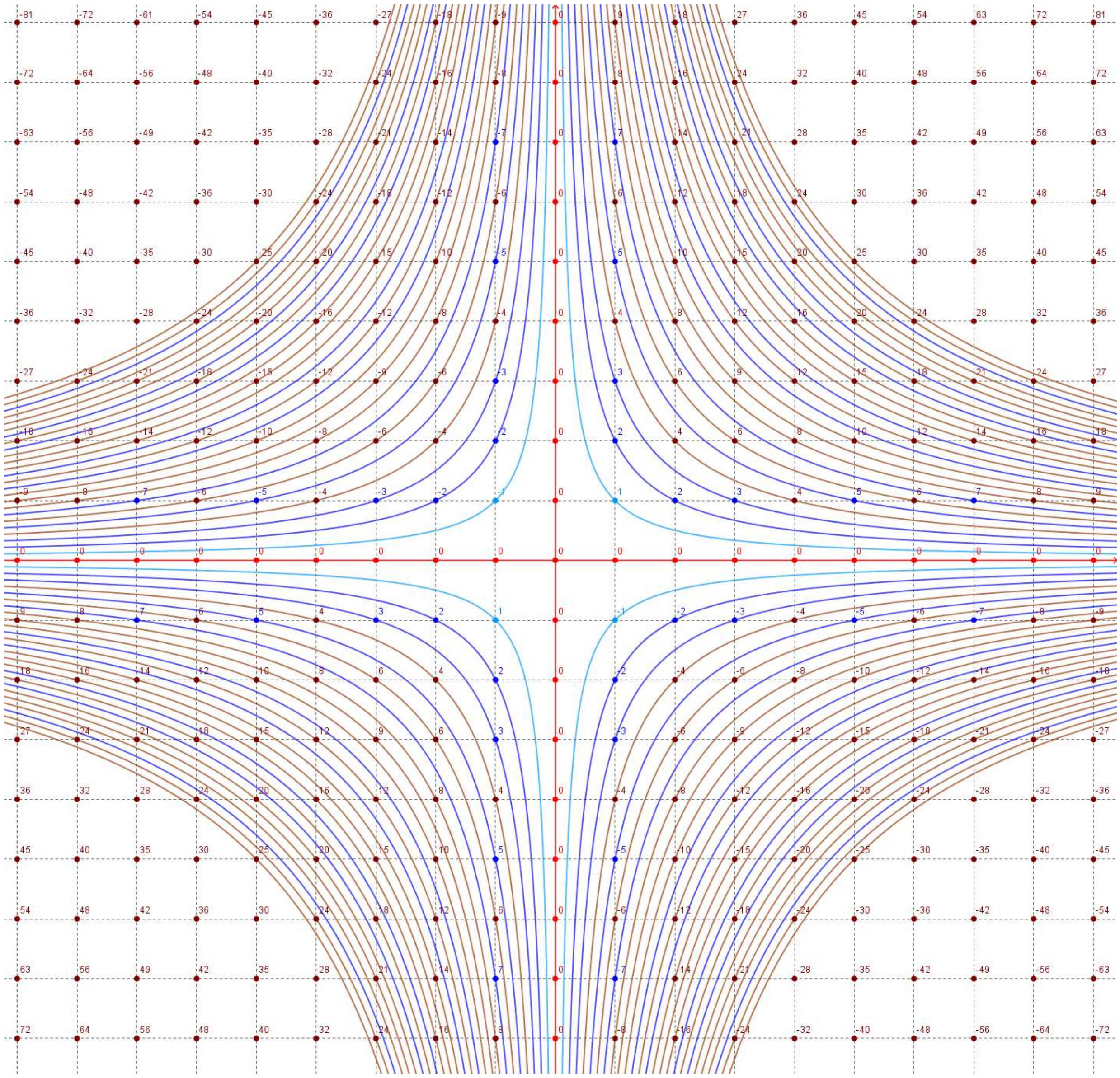


Figure 9. C000430 The hyperbolic lines in FMT.

Each point in the table is a result of the product $x * y$. Note that while the countless two-factor products form the hyperbolic lines, the 45° diagonals form the parabolic (quadratic) sequences of composites. They are in the form of $x = y(y \pm b)$.[8][9]

See below how FMT looks in the XY-plane.

Y-axis -->	15	-225	-210	-195	-180	-165	-150	-135	-120	-105	-90	-75	-60	-45	-30	-15	0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225
	14	-210	-196	-182	-168	-154	-140	-126	-112	-98	-84	-70	-56	-42	-28	-14	0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
	13	-195	-182	-169	-156	-143	-130	-117	-104	-91	-78	-65	-52	-39	-26	-13	0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
	12	-180	-168	-156	-144	-132	-120	-108	-96	-84	-72	-60	-48	-36	-24	-12	0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
	11	-165	-154	-143	-132	-121	-110	-99	-88	-77	-66	-55	-44	-33	-22	-11	0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
	10	-150	-140	-130	-120	-110	-100	-90	-80	-70	-60	-50	-40	-30	-20	-10	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
	9	-135	-126	-117	-108	-99	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
	8	-120	-112	-104	-96	-88	-80	-72	-64	-56	-48	-40	-32	-24	-16	-8	0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
	7	-105	-98	-91	-84	-77	-70	-63	-56	-49	-42	-35	-28	-21	-14	-7	0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
	6	-90	-84	-78	-72	-66	-60	-54	-48	-42	-36	-30	-24	-18	-12	-6	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
	5	-75	-70	-65	-60	-55	-50	-45	-40	-35	-30	-25	-20	-15	-10	-5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
	4	-60	-56	-52	-48	-44	-40	-36	-32	-28	-24	-20	-16	-12	-8	-4	0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
	3	-45	-42	-39	-36	-33	-30	-27	-24	-21	-18	-15	-12	-9	-6	-3	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
	2	-30	-28	-26	-24	-22	-20	-18	-16	-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
Y[1]	1	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Y[0]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Y[-1]	-1	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15
	-2	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	-22	-24	-26	-28	-30
	-3	45	42	39	36	33	30	27	24	21	18	15	12	9	6	3	0	-3	-6	-9	-12	-15	-18	-21	-24	-27	-30	-33	-36	-39	-42	-45
	-4	60	56	52	48	44	40	36	32	28	24	20	16	12	8	4	0	-4	-8	-12	-16	-20	-24	-28	-32	-36	-40	-44	-48	-52	-56	-60
	-5	75	70	65	60	55	50	45	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45	-50	-55	-60	-65	-70	-75
	-6	90	84	78	72	66	60	54	48	42	36	30	24	18	12	6	0	-6	-12	-18	-24	-30	-36	-42	-48	-54	-60	-66	-72	-78	-84	-90
	-7	105	98	91	84	77	70	63	56	49	42	35	28	21	14	7	0	-7	-14	-21	-28	-35	-42	-49	-56	-63	-70	-77	-84	-91	-98	-105
	-8	120	112	104	96	88	80	72	64	56	48	40	32	24	16	8	0	-8	-16	-24	-32	-40	-48	-56	-64	-72	-80	-88	-96	-104	-112	-120
	-9	135	126	117	108	99	90	81	72	63	54	45	36	27	18	9	0	-9	-18	-27	-36	-45	-54	-63	-72	-81	-90	-99	-108	-117	-126	-135
	-10	150	140	130	120	110	100	90	80	70	60	50	40	30	20	10	0	-10	-20	-30	-40	-50	-60	-70	-80	-90	-100	-110	-120	-130	-140	-150
	-11	165	154	143	132	121	110	99	88	77	66	55	44	33	22	11	0	-11	-22	-33	-44	-55	-66	-77	-88	-99	-110	-121	-132	-143	-154	-165
	-12	180	168	156	144	132	120	108	96	84	72	60	48	36	24	12	0	-12	-24	-36	-48	-60	-72	-84	-96	-108	-120	-132	-144	-156	-168	-180
	-13	195	182	169	156	143	130	117	104	91	78	65	52	39	26	13	0	-13	-26	-39	-52	-65	-78	-91	-104	-117	-130	-143	-156	-169	-182	-195
	-14	210	196	182	168	154	140	126	112	98	84	70	56	42	28	14	0	-14	-28	-42	-56	-70	-84	-98	-112	-126	-140	-154	-168	-182	-196	-210
	-15	225	210	195	180	165	150	135	120	105	90	75	60	45	30	15	0	-15	-30	-45	-60	-75	-90	-105	-120	-135	-150	-165	-180	-195	-210	-225
X-axis -->		-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Figure 10. C000430 The FMT in table form.

5. Definition of pair of complementary divisors.

Since the hyperbolic lines in FMT are always symmetric, then the same hyperbolic line that crosses $xy = 18$ in quadrant 1 will also cross 18 in quadrant 3. Symmetrically, the same hyperbolic line that crosses $xy = -18$ in quadrant 2 will also cross -18 in quadrant 4. This is true for all elements of the FMT.

Because of that, let's define the pair of complementary divisors from the FMT.

The origin of the idea of the pair of complementary divisors comes from the FMT – Full Multiplication table.

The notation of a pair of complementary divisors is $(d_1; d_2)$, where always $d_1 \leq d_2$.

Because of the symmetry in the FMT, we will always consider it to be just a single pair of complementary divisors when we exchange the sign of the two divisors of the pair. That is, below we have only one pair of complementary divisors:

$$(d_1; d_2) = (-d_2; -d_1)$$

For example, the number 18 has the following pairs of complementary divisors: $(1; 18), (2; 9), (3; 6)$. So, we say number 18 has three pairs of complementary divisors.

The central pair of complementary divisors is $(d_{c1}; d_{c2}) = (3; 6)$.

The trivial pair of complementary divisors is $(d_{t1}; d_{t2}) = (1; 18)$.

The number -18 has the following pairs of complementary divisors: $(-18; 1), (-9; 2), (-6; 3)$. So, we say number -18 has 3 pairs of complementary divisors.

The central pair of complementary divisors is $(d_{c1}; d_{c2}) = (-6; 3)$.

The trivial pair of complementary divisors is $(d_{t1}; d_{t2}) = (-18; 1)$.

There is no such pairs of complementary divisors in the form of $(-1; -18), (-2; -9), (-3; -6)$, nor in the form of $(18; 1), (9; 2), (6; 3)$ or $(18; -1), (9; -2), (6; -3)$.

d_1	The smallest complementary divisor $d_1 \leq \pm\sqrt{\lvert x \rvert} \leq d_2$, and $x = d_1 * d_2$.
d_2	The largest complementary divisor $d_2 \geq \pm\sqrt{\lvert x \rvert} \geq d_1$, and $x = d_1 * d_2$.
$(d_1; d_2)$	The pair of complementary divisors such that $d_2 \geq \pm\sqrt{\lvert x \rvert} \geq d_1$, $x = d_1 * d_2$.
d_{c1}	The smallest central complementary divisor $d_{c1} \leq \pm\sqrt{\lvert x \rvert} \leq d_{c2}$, and $x = d_{c1} * d_{c2}$. For positive divisors, d_{c1} is the sequence https://oeis.org/A033676.
d_{c2}	The largest central complementary divisor $d_{c2} \geq \pm\sqrt{\lvert x \rvert} \geq d_{c1}$, and $x = d_{c1} * d_{c2}$. For positive divisors, d_{c2} is the sequence https://oeis.org/A033677.
$(d_{c1}; d_{c2})$	The central pair of complementary divisors such that $d_{c2} \geq \pm\sqrt{\lvert x \rvert} \geq d_{c1}$, and $x = d_{c1} * d_{c2}$.
d_{t1}	The smallest trivial complementary divisor. $d_{t1} \leq \pm\sqrt{\lvert x \rvert} \leq 1 < d_{t2}$. Can be $-\lvert x \rvert$, -1, or 1.
d_{t2}	The largest trivial complementary divisor. $d_{t2} \geq \pm\sqrt{\lvert x \rvert} \geq -1 > d_{t1}$. Can be -1, 1, or $\lvert x \rvert$.
$(d_{t1}; d_{t2})$	The trivial pair of complementary divisors such that $d_{t2} \geq d_{t1}$ and $d_{t1} * d_{t2} = x = integer$. Then, because $d_{t1} \leq 1$, then d_{t1} can be $-\lvert x \rvert$, -1, or 1. Because $d_{t2} \geq -1$, d_{t2} can be -1, 1, or $\lvert x \rvert$.

The use of pair of complementary divisors justifies the equalities:

https://oeis.org/A161906 * https://oeis.org/A340791 = https://oeis.org/A340792
https://oeis.org/A319135 * https://oeis.org/A161908 = https://oeis.org/A340792
https://oeis.org/A027750 * https://oeis.org/A056538 = https://oeis.org/A061017

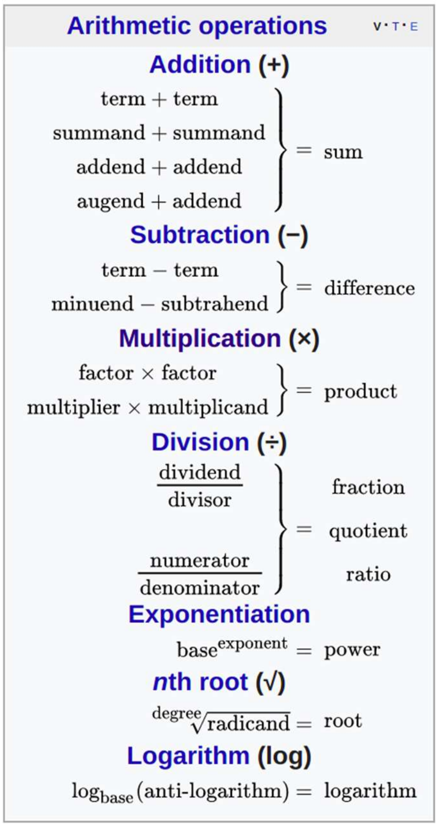
They depict what happens in the FMT – Full Multiplication Table.

5.1. Factor vs. Divisor (disambiguation).

Because of the discussion that took place in https://www.mersenneforum.org/showthread.php?t=28810 from post #33 https://www.mersenneforum.org/showpost.php?p=636399&postcount=33 forward, let us define what is a factor and what is a divisor of an integer.

Let us use the English language definitions mentioned in https://en.wikipedia.org/wiki/Multiplication and https://math.stackexchange.com/questions/4367999/for-commutative-arithmetic-why-do-we-have-asymmetrical-nomenclature-like-mul.

Here is the picture:



C002800 Figure 1 Reproduction of the figure used by Wikipedia about arithmetic operations available online at https://en.wikipedia.org/wiki/Multiplication.

Then we have:

$$factor1\ X\ factor2 = multiplier\ X\ multiplicand = product$$

$$\frac{dividend}{divisor} = \frac{numerator}{denominator} = fraction = quotient = ratio$$

Whenever we talk about factors or divisors, we are referring to integers. Like this:

$$factor1\ X\ factor2 = integer1\ X\ integer2$$

$$\frac{dividend}{divisor} = \frac{integer1}{integer2}$$

Then,

$$multiplier\ X\ multiplicand = complex\ number1\ X\ complex\ number2$$
$$\frac{numerator}{denominator} = \frac{complex\ number1}{complex\ number2}$$

That way, whenever we mention a product resulting from multiplication between integers, there will only be factors being multiplied.

Every product is the result of the multiplication of two or more factors. We won't say that a product is the result of multiplying n-divisors without mentioning that those n-divisors are now n-factors.

The only possibility of there being a divisor being multiplied by another number to result in a product is calculating the rest of the division.

Whenever we talk about integer division with rest 0, the integer denominator will always be a divisor. Never a factor. We won't say that an integer quotient with rest 0 is the result of dividend divided by a factor without mentioning that this factor is now a divisor.

5.2. The use of Factor vs. Divisor.

For a two factors multiplication to produce the desired integer, then the two factors to be multiplied necessarily need to be two complementary divisors.

Although in some cases the divisors of an integer are also the factors of it and vice-versa, this is not always true. For example, the composite number 6 has four positive divisors: $\{1,2,3,6\}$. But we cannot say that number 6 has 4 positive factors.

Number 36 may have four factors $36 = 1 * 2 * 3 * 6$, or three factors $36 = 1 * 6 * 6 = 1 * 3 * 12$, or two factors $36 = 1 * 36$. It becomes complex if we think of a definition of the number 36 using only the factors or only the multiplication.

But if we think only of the division, it is straight to the definition of any integer. Number 36 has exactly nine positive divisors: $\{1,2,3,4,6,9,12,18,36\}$. No other number has these exact 9 divisors set.

But if we transform these divisors into factors then we can produce several other products besides the number 36. We've lost control. That´s why the set of distinct positive and/or negative divisors completely and conclusively defines the primality of any integer.

Note how the use of distinct positive divisors from integer 36 avoids listing divisors 6 or 1 more than once.

When we talk about factors, repetitions of factors in composites mostly occur: $36 = 6 * 6 = 2 * 2 * 3 * 3 = 1 * 1 * 1 * ... * 1 * 36$.

The number of factors is not conclusive to define the primality of the integers. This is because we can always express any integer as each being the multiplication of only two factors. The multiplication table teaches us this.

The number of divisors is conclusive to define the primality of the integers. Especially when we use the concept of the pair of complementary divisors.

6. The TMTSP - Triangular Multiplication Table.

The TMT - Triangular Multiplication Table in the XY-plane:

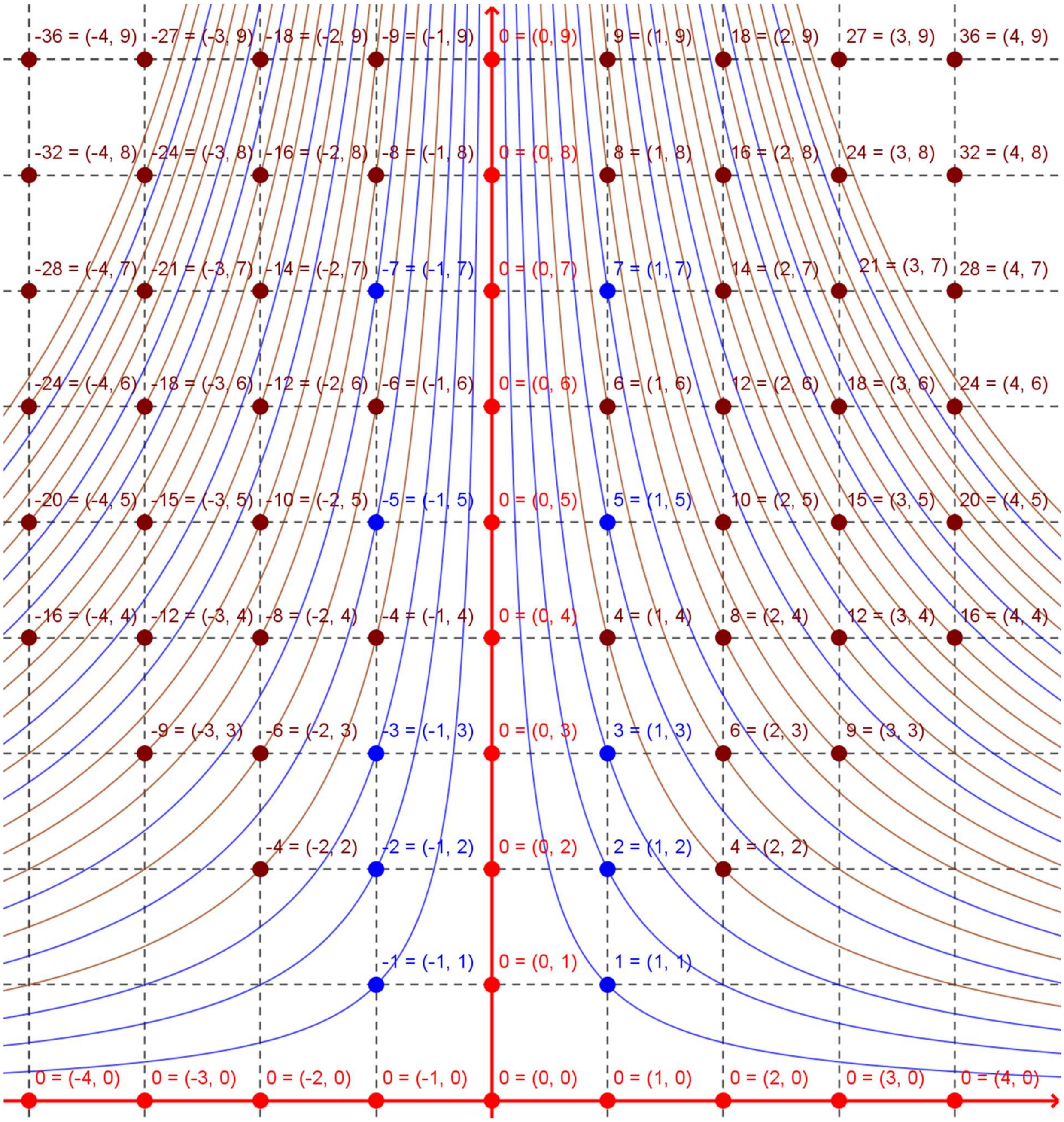


Figure 11. C001439 The hyperbolic lattice grid of the TMT - Triangular Multiplication Table.

The TMT is built in the 2nd and 3rd octets of the XY-plane. The 2nd octet occupies the triangular area of the 1st quadrant, and the 3rd octet occupies the adjacent triangular area in the 2nd quadrant.

The TMT - Triangular Multiplication Table reflects the pairs of complementary divisors for all integers.

The first use of the name TMT - Triangular Multiplication Table in OEIS is dated Nov 08, 2001, at sequences https://oeis.org/A062858 and https://oeis.org/A062859.

6.1. The TMTSP - Triangular Multiplication Table Sieve of Primes.

The Triangular Multiplication Table Sieve of Primes:

- The point $(x, y) = (\sqrt{0}, \sqrt{0})$ in the diagonal line transverse axis $x = y$ is the integer 0. Because the hyperbola $HL[x, y] = xy = 0$ intersect with infinitely many other points with integer XY-coordinates besides (0,0), then number 0 is a composite number.
- The point $(x, y) = (\sqrt{1}, \sqrt{1})$ in the diagonal line transverse axis $x = y$ is the integer 1. Because the hyperbola $HL[x, y] = xy = 1$ does not intersect with any other point with integer XY-coordinates besides $(1,1)$, then number 1 is a prime number.
- The point $(x, y) = (\sqrt{2}, \sqrt{2})$ in the diagonal line transverse axis $x = y$ is the integer 2. Because the hyperbola $HL[x, y] = xy = 2$ does not intersect with any other point with integer XY-coordinates besides $(1,2)$, then number 2 is a prime number.
- The point $(x, y) = (\sqrt{3}, \sqrt{3})$ in the diagonal line transverse axis $x = y$ is the integer 3. Because the hyperbola $HL[x, y] = xy = 3$ does not intersect with any other point with integer XY-coordinates besides $(1,3)$, then number 3 is a prime number.
- The point $(x, y) = (\sqrt{4}, \sqrt{4})$ in the diagonal line transverse axis $x = y$ is the integer 4. Because the hyperbola $HL[x, y] = xy = 4$ does intersect at least one another point with integer XY-coordinates besides (1,4), then number 4 is a composite number.
- The point $(x, y) = (\sqrt{5}, \sqrt{5})$ in the diagonal line transverse axis $x = y$ is the integer 5. Because the hyperbola $HL[x, y] = xy = 5$ does not intersect with any other point with integer XY-coordinates besides $(1,5)$, then number 5 is a prime number.
- The point $(x, y) = (\sqrt{6}, \sqrt{6})$ in the diagonal line transverse axis $x = y$ is the integer 6. Because the hyperbola $HL[x, y] = xy = 6$ does intersect at least one another point with integer XY-coordinates besides (1,6), then number 6 is a composite number.
- The point $(x, y) = (\sqrt{7}, \sqrt{7})$ in the diagonal line transverse axis $x = y$ is the integer 7. Because the hyperbola $HL[x, y] = xy = 7$ does not intersect with any other point with integer XY-coordinates besides $(1,7)$, then number 7 is a prime number.

- The point $(x, y) = (\sqrt{8}, \sqrt{8})$ in the diagonal line transverse axis $x = y$ is the integer 8. Because the hyperbola $HL[x, y] = xy = 8$ does intersect at least one another point with integer XY-coordinates besides (1,8), then number 8 is a composite number.
- The point $(x, y) = (\sqrt{9}, \sqrt{9})$ in the diagonal line transverse axis $x = y$ is the integer 9. Because the hyperbola $HL[x, y] = xy = 9$ does intersect at least one another point with integer XY-coordinates besides (1,9), then number 9 is a composite number.
- And so on…

7. Conclusion.

The results obtained for the primality of integers in SMTSP and TMTSP are exactly the same. They follow the properties of the hyperbolic lattice-grid.

Each sieve can be called a “hyperbolic sieve". They present the same results found in the “parabolic sieve” discovered by Matiyasevich, Yuri and Stechkin, Boris. (1999) "A visual Sieve for prime Numbers", available online at https://logic.pdmi.ras.ru/~yumat/personaljournal/sieve/sieve.gif.

Funding.

This research received no external funding.

Conflicts of Interest.

The author declare no conflicts of interest.

Acknowledgments.

I would like to thank all the essential support and inspiration provided by Mr. H. Bli Shem and my Family. A Talmud proverb says, “He who does not teach his son a trade teaches him to steal”. The wisest of trades is study. I dedicate these studies to my children.

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