A Topological Approach to Consumer Theory

Manuel Alberto M. Ferreira

May 8, 2022

A Topological Approach to Consumer Theory

Manuel Alberto M. Ferreira

Instituto Universitário de Lisboa (ISCTE-IUL), BRU-IUL, ISTAR-IUL, Lisboa

PORTUGAL

manuel.ferreira@iscte.pt

ABSTRACT

The presentation of an interesting application of mathematics in economics, more concretely of topological notions in economics: the formulation of the theory of consumer basic problem built on topology, is the main target of this work. This is made grounded on the concept of preferences relation and operationalized with optimization tools.

Keywords: *Preferences relation, optimization, consumer theory, demand.*

INTRODUCTION

This work target is to present a mathematical formulation of the consumer theory,

as rigorous as possible, using mostly topological notions. This is done using mainly the concept of preferences relation and then optimization mathematical tools.

So, in the next section, the preferences relation concept and the respective axioms are presented and the model to be used is outlined.

Then it is established the consumer theory and some important consequent results are shown.

Finally, the paper ends with a section of conclusions and bibliography on these subjects.

INTRODUCING PREFERENCES RELATIONS

To use mathematical tools to deal with the concept of consumer basket, consider a non-empty convex closed subset X in ${\mathbb{R}_+}^n$ where is defined a preferences relation, symbolized $\succcurlyeq$, as follows:

Definition 2.1

For any x, y and z belonging to X,

i) $x \succcurlyeq y \vee y \succcurlyeq x$ (Completeness), *ii*) $x \succcurlyeq y \ \wedge \ y \succcurlyeq z \ \Longrightarrow \ x \succcurlyeq z$ (Transitivity)

iii) For any $y \in X$, $\{x \in X: x \succcurlyeq y\}$ and $\{x \in X: y \succcurlyeq x\}$ are closed subsets (Continuity),

iv) $x \geq y$ (that is:$x_i \geq y_i, i = 1,2,..n$) and $x \neq y \Longrightarrow x \succ y$ (Strong Monotonicity).

Note:

- $x \succ y$ if $x \succcurlyeq y$ and $y \succcurlyeq x$ is false,

- In economics language X is the set of a consumer possible baskets and $\succcurlyeq$

represents that consumer preferences lengthwise this set of baskets.

Definition 2.2

$x, y \in X$ are indifferent, and it is denoted $x \sim y$, when $x \succcurlyeq y$ and $y \succcurlyeq x$.

Definition 2.3[1]

A basket $x \in X$ is said to be redundant when there is $y \in X$ such that $x \geq y, y \neq x,$ and $x \sim y$.

To present some results, to the Definition 2.1 must be added the following two axioms:

v) $x, y \in X, x \neq y$ and x, y not redundant such that $x \sim y \Longrightarrow (1-t)x + ty \succ x, \forall t \in [0,1]$.

Note:

-That is: the convex linear combination of two different, not redundant, and indifferent baskets is preferred to one of them isolated.

vi) $\forall x \in X, \forall \varepsilon > 0, \exists y \in X$ such that $\|x - y\| < \varepsilon$ and $y \succ x$.

Note:

-That is: in any neighborhood of a basket there is at least one other basket preferable to it.

Theorem 2.1

Supposing that $\succcurlyeq$ fulfills the axioms *i), ii), iii)* and *iv)* in Definition 2.1, if $X = \mathbb{R}_+{}^n$ there is a continuous function $u: X \to \mathbb{R}$ such that $u(x) \geq u(y)$ if and only if $x \succcurlyeq y, \forall x, y \in X$.

Dem: Call $\mathbf{1}$ the vector with the whole n coordinates equal to 1. For $x \in X$ define $A = \{t \in \mathbb{R}_+: t\mathbf{1} \succcurlyeq x\}$ and $B = \{t \in \mathbb{R}_+: x \succcurlyeq t\mathbf{1}\}$. Per *iv)* A and B are nonempty and by *iii)* are

[1]See Simonsen (1989).

both closed since the function $\varphi: \begin{matrix} \mathbb{R}_+ \to \mathbb{R}_+{}^n \\ t \to t\mathbf{1} \end{matrix}$ is continuous and, here,

$A = \varphi^{-1}(\{y \in X: y \succcurlyeq x\})$ and $B = \varphi^{-1}(\{y \in X: x \succcurlyeq y\})$. After *i)* it results $A \cup B = \mathbb{R}_+$and by the $\mathbb{R}_+$ connexity [2] there is $t \in \mathbb{R}$ fulfilling $t\mathbf{1}$~x. Again, by *iv)* this *t* is unique. So define $u: X \to \mathbb{R}$ as $u(x) = t$. Besides $u^{-1}([t_0, \infty]) = \{x \in X: u(x) \geq t_0\} = \{x \in X: x \succcurlyeq u^{-1}(t_0)\}$ and $u^{-1}([0, t_0]) = \{x \in X: u^{-1}(t_0) \succcurlyeq x\}$ are closed and then it results that *u* is continuous[3].□

Note:

-More generally it is also true that "If $X \subset \mathbb{R}^n$is a connex set and $\succcurlyeq$ is a preferences relation in *X* fulfilling *i), ii)* and *iii)*, there is a continuous function $u: X \to \mathbb{R}$ such that $u(x) \geq u(y) \Leftrightarrow x \succcurlyeq y, \forall x, y \in X$", see Debreu (1959),

-This theorem shows that under axioms *i), ii), iii)* and *iv)* it is possible to determine a numerical scale for the consumer preferences what is extremely important in Decision Theory.

ESTABLISHING CONSUMER THEORY

Begin by a formulation, among many equivalents, of the theory of consumer basic problem:

-Call *r* the consumer income and $p = (p_1, p_2, \ldots, p_n)$ the *n* goods vector prices. If $u: X \to \mathbb{R}$ represents the consumer preferences, the preferences maximization problem can be mathematically formulated as

$$\begin{matrix} \max \ u(x) \\ s.to \ \langle p, x \rangle \leq r \end{matrix} \quad (3.1).$$

It is supposed that

$$A = \{x \in X: \langle p, x \rangle \leq r\},$$

the opportunities set, is non-empty.

Note:

-Being u continuous and $p \in \mathbb{R}^n_{++}$, so being A compact, the Weierstrass theorem guaranties that (3.1) is possible.

[2] A set is connected if it cannot be represented as the union of two separable sets both non-empty.
[3] Because in this case the whole closed sets in $[0, \infty)$ will have inverse image closed. For details see (Cysne and Moreira, 1997), page 95.

Definition 3.1

The indirect utility function $v(p,r)$ is defined through the problem:

$$\begin{array}{r} v(p,r) = \max\ u(x) \\ s.to\ \langle p,x\rangle \le r, x \in X \end{array} \quad (3.2).$$

Note:

-It is easy to check that if *x* is a solution for (3.2) $\langle p,x\rangle = r$ since either *vi)* or *iv)* are satisfied for $X = \mathbb{R}_+{}^n$,

- *x* is not redundant,

- If, in addition, *v)* is satisfied *x* is unique.

Definition 3.2

The vector only solution *x* for the problem (3.2), given *p* and r: $x(p,r)$ is the Marshallian demand.

Note:

-$x_i(p,r)$ is the Marshallian demand for the good $i, i = 1,2,\dots,n$.

Theorem 3.1

If consumer preferences satisfy *i)*, *ii)*, *iii)* and *vi)*, the Marshallian demand $x_i\colon \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous $\forall i = 1,2,\dots,n$.

Dem (See Simonsen (1989)): Be $(p_n, r_n) \in \mathbb{R}^{n+1}_{++}$ and $(p_n, r_n) \longrightarrow (p,r) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$.Being easy to check that the sequence $\left(x(p_n,r_n)\right)_{n\in\mathbb{N}}$ is bounded, to evidence that the Marshallian demand is continuous, it is enough to evidence that any $\left(x(p_n,r_n)\right)_{n\in\mathbb{N}}$ convergent subsequence converges to $x(p,r)$. So consider the convergent subsequence $\left(x\left(p_{n_i}, r_{n_i}\right)\right)_{i\in\mathbb{N}}$ converging to *y*. As $\langle p_{n_i}, x\left(p_{n_i}, r_{n_i}\right)\rangle = r_{n_i}$, in the limit $\langle p,y\rangle = r$, that is: y is feasible in what respects (p,r). To evidence that $y = x(p,r)$ it is enough to evidence that if y' is feasible in what respects $(p,r), u(y) \ge u(y')$. Then define $\mu_n = \frac{\langle p,y'\rangle}{\langle p_n,y'\rangle}\frac{r_n}{r}$ .It is immediate to check that $\langle p_n,\ \mu_n y'\rangle = \frac{\langle p,y'\rangle}{r} r_n \le r_n$. This means that $\mu_n y'$ is feasible in what respects (p_n, r_n). So, as $x(p_n, r_n)$ is the consumer equilibrium point in what respects the pair (p_n, r_n), $u\left(x(p_n,r_n)\right) \ge u(\ \mu_n y')$ and so going to the limit when $i \to \infty$, remarking that $\mu_n \to 1$ when $n \to \infty$, it is obtained $u(y) \ge u(y')$ so completing the proof.□

CONCLUDING REMARKS

In this paper, economic concepts were presented through a rigorous formulation, allowing that some economic results were demonstrated in the mathematical sense.
Plays here a fundamental role the concept of preferences relation that is axiomatically formulated.

BIBLIOGRAPHY

1. BORCH, K. (1974), *The Mathematical Theory of Insurance*. Lexington, Mass.: Lexington Books, D.C. Heath and Co.
2. BORCH; K. (1990), *Economics of Insurance*. Edited by Aase K. K. and Sandmo A., Amsterdam, North-Holland.
3. CHAVAGLIA, J., FILIPE, J. A., FERREIRA, M. A. M. (2016), ”Neuroeconomics and reinforcement learning an exploratory analysis”, *APLIMAT 2016-15th Conference on Applied Mathematics 2016, Proceedings 2016*, Bratislava, 527-534.
4. CYSNE, R. P., MOREIRA, H. A. (1997), *Curso de Matemática para Economistas*. São Paulo, Editora Atlas, S. A..
5. DEBREU, G. (1959), *Theory of value and axiomatic analysis of economic equilibrium*. Monograph, 17, Cowles Foundation.
6. FERREIRA, M. A. M. (2015), “The Hahn-Banach theorem in convex programming”, *APLIMAT 2015-14th Conference on Applied Mathematics 2015, Proceedings 2015,* Bratislava, 283-291.
7. FERREIRA, M. A. M. (2015a), “The minimax theorem as Hahn-Banach theorem consequence”, *Acta Scientiae et Intellectus*, 1 (1), 58-65.
8. FERREIRA, M. A. M., AMARAL, I. (2002), *Cálculo Diferencial em* $\mathbb{R}^n$. Edições Sílabo-5ª Edição, Lisboa.
9. FERREIRA, M. A. M., AMARAL, I. (2005), *Integrais Múltiplos e Equações Diferenciais*. Edições Sílabo-5ª Edição, Lisboa.
10. FERREIRA, M. A. M., FILIPE, J. A. (2014), “Convex sets strict separation in Hilbert spaces”, *Applied Mathematical Sciences,* (61-64), 3155-3160. **DOI**: 10.12988/ams.2014.44257
11. FERREIRA, M. A. M., FILIPE, J. A. (2015), “A ratio to evaluate harvest procedures management in an economic system where resources dynamics is ruled by an Ornstein-Uhlenbeck process”, *Applied Mathematical Sciences,* 9(25-28), 1213-1219. **DOI**: 10.12988/ams.2015. 410871
12. FERREIRA, M. A. M., MATOS, M. C. (2014), “Convex sets strict separation in the minimax theorem”, *Applied Mathematical Sciences*, 8(33-36), 1781-1787. **DOI**: 10.12988/ams.2014.4271
13. FERREIRA, M. A. M., ANDRADE, M., MATOS, M. C., FILIPE, J. A., COELHO, M. (2012), “Kuhn-Tucker’s Theorem-the Fundamental Result in Convex Programming Applied to Finance and Economic Sciences”, *International Journal of Latest Trends in Finance & Economic Sciences*, 2 (2), 111-116.
14. FILIPE, J. A., FERREIRA, M. A. M. (2013), “Chaos in humanities and social sciences: An approach”, *APLIMAT 2013-12th Conference on Applied Mathematics 2013, Proceedings 2013,* Bratislava.

15. NEUMÄRKER, B. (2007), "Neuroeconomics and the Economic Logic of Behavior", *Analyse & Kritik*, 29, 60-85.
16. ROBERTS, B., SCHULZE, D. L. (1973), *Modern Mathematics and Economic Analysis*. New York: W. W. Norton & Company, Inc..
17. SIMONSEN, M. H. (1989), *Análise convexa no* $\mathbb{R}^n$. Rio de Janeiro: EPGE, nº 145.