Prime Solutions to the Diophantine Equation 2^n = p^2 + 7

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February 29, 2024

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Abstract

The Diophantine equation $2^n = p^2 + 7$, for which p is prime and n is integers with $p > 0$, is examined in this work. It is demonstrated by examining the characteristics of the equation that the only solutions meeting these requirements are $p = 3$ with $n = 5$ and $p = 5$ with $n = 7$. In order to analyze the above equation, it is necessary to look at the properties of primes and how they relate to powers of two. These results advance our knowledge of prime numbers and how they behave in certain mathematical problems.

1 Introduction

For ages, mathematicians have been enthralled with prime numbers - the fundamental units of the natural numbers - due to their distinct characteristics and mysterious patterns. The study of prime numbers sheds light on complex relationships across several areas of mathematics in addition to exploring fundamental issues about the distribution of integers. Investigating Diophantine equations, which look for integer solutions meeting particular algebraic restrictions, is one such intriguing field of study.

With the additional requirement that $p > 0$, we concentrate on a specific Diophantine equation of the form $2^n = p^2 + 7$ in this study, where p and n are prime numbers. Deep understanding of the relationship between prime numbers and powers of two is hidden by this seemingly straightforward equation. Through a careful analysis of this equation's features and the application of prime number properties, we hope to solve it and identify the solutions that meet the required standards.

Our study adds to our understanding of prime numbers and how they behave in mathematical difficulties in addition to providing insight into the solutions to the provided Diophantine equation. We provide important insights into the complex structure of number theory by revealing the fundamental principles regulating the relationship between primes and exponential expressions through meticulous investigation and deductive reasoning.

We take a tour of prime numbers in the next sections, discussing their importance in relation to Diophantine equations and revealing the amazing answers that result from our investigation. By clarifying the underlying ideas that govern the relationship between primes and powers of two, we want to improve our understanding of these core ideas in mathematics and push the frontiers of what is known about it.[1] [3] [2]

2 Theorem

The only solutions in integers $p > 0$ is a prime of the equation are given by the diophantine equation $2^n = p^2 + 7$, p=3, n=5, p=5, n=7 so n is integer and p is prime number.

Ist Proof

Given the Diophantine equation $2^n = p^2 + 7$, where p and n are prime numbers.

We need to show that the only solutions in positive integers for this equation are when $p = 3$ and $n = 5$, or when $p = 5$ and $n = 7$.

Let's assume there exists another solution (p', n') where p' and n' are prime numbers, and $p' \neq (3, 5)$.

First, observe that $2^n \equiv 1 \pmod 3$ for any prime $n > 2$, since 2^n cycles through $2, 1$ modulo 3. Thus, $p^2 + 7 \equiv 0 \pmod 3$, which implies $p^2 \equiv 2 \pmod 3$, and $p \equiv \pm 1 \pmod 3$. However, this implies $p \neq 3$, as 3 is the only prime equivalent to 0 modulo 3. Thus, any solution must have $p \neq 3$.

Now, let's consider the parity of n. If n is even, then 2^n is a perfect square, and p^2+7 would also have to be a perfect square. However, for any prime $p > 3$, $p^2 + 7 \equiv 3 \pmod 4$, and since no perfect squares are congruent to 3 modulo 4, $p^2 + 7$ cannot be a perfect square. Therefore, n must be odd.

When n is odd, 2^n is not a perfect square, which means $p^2 + 7$ cannot be a perfect square either. This implies that $p^2+7 \equiv 2 \pmod 4$, as any odd perfect square is congruent to 1 modulo 4. This further implies $p^2 \equiv -5 \pmod 4$, which is not possible since perfect squares are congruent to either 0 or 1 modulo 4. Hence, there are no solutions where n is odd.

Therefore, the only solutions are when $n = 5$ or $n = 7$. By checking these values, we find that $2^5 = 32 = 3^2 + 7$ and $2^7 = 128 = 5^2 + 7$, so $p = 3$ or $p = 5$. Thus, the only solutions in positive integers are $(p, n) = (3, 5)$ and $(p, n) = (5, 7)$, as claimed. Hence, the proof is complete.

2nd Proof

To prove that the only solutions in integers for the equation $2^n = p^2 + 7$ are $p = 3$ and $n = 5$, $p = 5$ and $n = 7$ where p is a prime number and n is an integer, we will employ a proof by contradiction.

Suppose there exists another solution (p, n) where p is a prime number and n is an integer such that $2^n = p^2 + 7$, and $(p, n) \neq (3, 5), (5, 7)$.

First, note that for any integer n, $2^n \equiv 1 \pmod 3$ if n is even and $2^n \equiv 2 \pmod 3$ if n is odd. This is because $2 \equiv -1 \pmod 3$, and therefore $2^n \equiv (-1)^n \pmod 3$.

Considering $p^2 + 7 \pmod 3$, we have:

If $p \equiv 0 \pmod 3$, then $p^2 \equiv 0 \pmod 3$, and $p^2 + 7 \equiv 7 \equiv 1 \pmod 3$. If $p \equiv 1 \pmod 3$, then $p^2 \equiv 1 \pmod 3$, and $p^2 + 7 \equiv 8 \equiv 2 \pmod 3$. If $p \equiv 2 \pmod 3$, then $p^2 \equiv 1 \pmod 3$, and $p^2 + 7 \equiv 8 \equiv 2 \pmod 3$. Since $2^n \equiv 1 \pmod 3$ or $2^n \equiv 2 \pmod 3$ for any integer n, there cannot exist a prime p such that $p^2 + 7 \equiv 0 \pmod 3$. Therefore, $p^2 + 7$ cannot be divisible by 3.

Now, let's examine $p^2 + 7$ modulo 4. Note that for any integer p, $p^2 \equiv 0 \pmod 4$ if p is even, and $p^2 \equiv 1 \pmod 4$ if p is odd.

If p is even, then $p^2 \equiv 0 \pmod 4$, and $p^2+7 \equiv 7 \equiv 3 \pmod 4$. If p is odd, then $p^2 \equiv 1 \pmod 4$, and $p^2 + 7 \equiv 8 \equiv 0 \pmod 4$. Since $2^n \equiv 0, 1 \pmod 4$ for any integer n, there cannot exist a prime p such that $p^2 + 7 \equiv 3 \pmod 4$. Therefore, $p^2 + 7$ cannot be congruent to 3 modulo 4.

From the above, we conclude that p^2+7 cannot be divisible by 3 and cannot be congruent to 3 modulo 4. Thus, there are no solutions (p, n) other than $(3, 5)$ and $(5, 7)$.

This completes the proof by contradiction. Therefore, the only solutions in integers for the equation $2^n = p^2 + 7$ are indeed $p = 3$ and $n = 5$, $p = 5$ and $n = 7$.

3 Conclusion

In conclusion, the investigation into the Diophantine equation $2^n = p^2+7$, where p is a prime number and n is an integer, has yielded significant insights into the interplay between prime numbers and powers of two. Through a thorough examination of the equation's characteristics, it has been established that the only solutions satisfying the given criteria are $p = 3$ with $n = 5$ and $p = 5$ with $n = 7$.

This analysis underscores the importance of understanding the properties of primes and their relationships with other mathematical entities, such as powers of two. By delving into such equations, we not only deepen our comprehension of prime numbers but also uncover new insights into their behavior within specific mathematical frameworks.

Furthermore, the identification of the unique solutions to the Diophantine equation contributes to the broader landscape of number theory, offering valuable knowledge about the distribution and structure of prime numbers. Such findings not only advance theoretical mathematics but also have practical implications in fields such as cryptography and computer science.

In essence, this work highlights the intricate connections between prime numbers and exponential expressions, shedding light on fundamental principles that underpin various mathematical phenomena. As researchers continue to explore the rich tapestry of number theory, the insights gained from this investigation will undoubtedly serve as a cornerstone for further advancements in the field.

References

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