EasyChair Preprint

№ 10296

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October 25, 2023

MORGAN-STONE LATTICES

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Abstract. *Morgan-Stone (MS) lattices* are axiomatized by the constant-free identities of those axiomatizing *Morgan-Stone (MS) algebras*. Applying the technique of characteristic functions of prime filters as homomorphisms from lattices onto the two-element chain one and their products, we prove that the variety of MS lattices is the abstract hereditary multiplicative class generated by a six-element one with an equational disjunctive system expanding the direct product of the three- and two-element chain distributive lattices, in which case subdirectly-irreducible MS lattices are exactly isomorphic copies of nine non-one-element pair-wise non-isomorphic subalgebras of the six-element generating MS lattice, and so we get a 29-element non-chain distributive lattice of varieties of MS lattices subsuming the four-/three-element chain one of "De Morgan"/Stone lattices/algebras (viz., constant-free versions of De Morgan algebras)/(more precisely, their term-wise definitionally equivalent constant-free versions, called *Stone lattices*). Among other things, we provide an REDPC scheme for MS lattices. Laying a special emphasis onto the universal/[quasi-]equational unbounded approximation of MS algebras (viz., the greatest universal/[quasi-]equational class of MS lattices without members with both bounds but expandable to no MS algebra), we find a 29-element non-chain distributive lattice of its sub-quasi-varieties, subsuming the fifteen-element one of the [quasi-]equational join (viz., the [quasi-]variety generated by the union) of De Morgan and Stone lattices, in its turn, subsuming the eight-element one of those of the variety of De Morgan lattices found earlier, each of the rest being the quasi-equational join of its intersection with the variety of De Morgan lattices and the variety of Stone lattices, as well as provide a complete description of their relative subdirectly-irreducibles resulting from REDPC for MS lattices. In this connection, we also prove that relatively simple quasi-varieties of MS lattices/algebras are exactly varieties of almost/ De Morgan lattices/algebras, the reservation "almost" meaning presence of subdirectly-irreducibles not expandable to MS algebras, in which case there is a strictly decreasing countable chain of quasi-varieties of almost De Morgan lattices containing all De Morgan lattices, and so its intersection is not finitely axiomatizable, though the lattice of quasi-vaarieties of almost Kleene lattices is proved to be finite but not distributive.

1. Introduction

The notion of *De Morgan lattice*, being originally due to [16], has been independently explored in [11] under the term *distributive i-lattice* w.r.t. their subdirectly-irreducibles and the lattice of varieties. They satisfy so-called *De Morgan identities*. On the other hand, these are equally satisfied in *Stone algebras* (cf., e.g., [8]). This has inevitably raised the issue of unifying such varieties. Perhaps, a first way of doing it within the framework of De Morgan algebras (viz., bounded De Morgan lattices; cf., e.g., [2]) has been due to [3] (cf. [24]) under the term *Morgan-Stone (MS) algebra* providing a description of their subdirectly-irreducibles, among which there are those being neither De Morgan nor Stone algebras. Here, we study unbounded MS algebras naturally called *Morgan-Stone (MS) lattices*. Demonstrating the usefulness of the technique of the characteristic functions of prime filters and

2020 *Mathematics Subject Classification.* 06D15, 06D30, 08A30, 08B05, 08B26, 08C15.
Key words and phrases. De Morgan lattice, Stone algebra, quasi-variety, REDPC..

functional products of former ones as well as disjunctive systems, we briefly discuss the issues of subdirectly-irreducible Morgan-Stone lattices and their varieties. Likewise, summarizing construction of REDPC schemes (cf. [7]) for distributive lattice[expansion]s originally being due to [9] [and [13, 22]], we provide that for Morgan-Stone lattices and an enhanced one for the {quasi-}equational join of De Morgan and Stone lattices. Nevertheless, the culminating issue of this study is to find the lattice of sub-quasi-varieties of the equational unbounded approximation of MS algebras upon the basis of that of the variety of De Morgan lattices found in [18]. In this connection, we also prove that relatively simple quasi-varieties of MS lattices/algebras are exactly varieties of almost/ De Morgan lattices/algebras, while the equational unbounded approximation of MS algebras is equally the universal one, whereas there is a strictly decreasing countable chain of quasi-varieties of almost De Morgan lattices subsuming all De Morgan lattices, its intersection being then not finitely axiomatizable, though the lattice of quasi-vaarieties of almost Kleene lattices is proved to be finite but not distributive.

Perhaps, the principal advance of this study with regard to [18] consists in not merely extending it beyond De Morgan lattices but mainly in providing complete description of relative subdirectly-irreducibles of the finitely-generated quasi-varieties of MS lattices under consideration, essentially based upon REDPC for MS lattices as well as congruence decomposition (in its turn, going back to congruence ideality [7]) for varieties with REDPC, thus once more demonstrating the power of the ideas underlying the outstanding work [7].

In general, we seek to expand our results to bounded MS lattices properly subsuming MS algebras, whenever it is at all possible. This equally concerns the issues of subdirectly-irreducibles and the lattice of varieties but not the one of quasi-varieties because of the well-known infiniteness (more specifically, Q-universality) of that of De Morgan algebras [1].

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Then, in Section 3 we briefly summarize general issues concerning REDPC in the sense of [7] as well as equational implicative/disjunctive systems in the sense of [21]/[20] in connection with simplicity/"subdirect irreducibility". Next, Section 4 is devoted to preliminary study of Morgan-Stone lattices as for their generating algebra, subdirectly-irreducibles and the lattice of varieties. Further, Section 5 is a thorough collection of culminating results on sub-quasi-varieties of the equational unbounded approximation of Morgan-Stone algebras. Likewise, Section 6 is devoted to characterizing relatively semi-simple quasi-varieties of MS lattices/algebras. Finally, Section 7 is a concise collection of open issues.

2. General background

2.1. **Set-theoretical background.** Non-negative integers are identified with the sets/ordinals of lesser ones, "their set/ordinal"|"the ordinal‖set class" being denoted by $\omega|(\infty\|\Upsilon)$. Unless any confusion is possible, one-element sets are identified with their elements. To avoid any confusion because of the standard identification of Υ^n (viz., the class of functions with domain $n \in \omega$) with $[\Upsilon^{n-1}\times]\Upsilon$, when $n = (1[+m])$ [where $m \in \omega$], the projection operator/function π is endowed with the superscript specifying the domain of arguments *only in such cases* {but not in general}.

For any sets A, B and D as well as $\theta \subseteq A^2$, $h : A \to B$ and $g : A^2 \to A$, let $\wp_{[K]}((B,)A)$ be the set of all subsets of A (including B) [of cardinality in $K \subseteq \infty$, $D \subseteq_K A$ standing for $D \in \wp_K(A)$], $((\Delta_A|\nu_\theta)\|(A/\theta)\|\chi^B_A) \triangleq (\{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}\|\nu_\theta[A]\|(((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\})))$, $A^{*|+} \triangleq (\bigcup_{m\in(\omega\setminus(0|1))} A^m)$,

$h_* : A^* \to B^* : a \mapsto (a \circ h)$, $g_+ : A^+ \to A, \langle [\langle a,b\rangle,]c\rangle \mapsto [g]([g_+(\langle a,b\rangle),]c)$ and $\varepsilon_B : (\Upsilon^B)^2 \to \wp(B), \langle d,e\rangle \mapsto \{b \in B \mid \pi_b(d) = \pi_b(e)\}$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, for any $(\bar{a}|C) \in (A^*|\wp(A))$, by induction on the length (viz., domain) of any $\bar{b} = \langle [\bar{c}, d]\rangle \in A^*$, put $((\bar{a} * \bar{b})|(\bar{b}(\cap/\backslash)C)) \triangleq (([\langle]\bar{a}[*\bar{c}, d\rangle])|(\langle [\bar{c}(\cap/\backslash)C(,d)]\rangle))$ |[(provided $d \in / \notin C$)]. Likewise, given any $\overline{S} \in \wp(D)^B$ and $\bar{f} \in \prod_{b\in B} S_b^A$, we have its *functional product* $(\prod^{\mathrm{F}} \bar{f}) : A \to (\prod_{b\in B} S_b), a \mapsto \langle f_b(a)\rangle_{b\in B}$ such that

$$\ker(\prod\nolimits^{\mathrm{F}} \bar{f}) \quad = \quad (A^2 \cap (\bigcap_{b\in B} (\ker f_b))), \tag{2.1}$$

$$\forall b \in B : f_b \quad = \quad ((\prod\nolimits^{\mathrm{F}} \bar{f}) \circ \pi_b), \tag{2.2}$$

$f_0 \odot f_1$ standing for $(\prod^{\mathrm{F}} \bar{f})$, whenever $B = 2$.

A *lower/upper cone of* a poset $\mathcal{P} = \langle P, \leqq\rangle$ is any $C \subseteq P$ such that, for all $a \in C$ and $b \in P$, $(a \geqq / \leqq b) \Rightarrow (b \in C)$. Then, an $a \in S \subseteq P$ is said to be *minimal/maximal in* S, if $\{a\}$ is a lower/upper cone of S, their set being denoted by $(\min / \max)_{\mathcal{P}|\leqq}(S)$, in case of the equality of which to S, this being called an *anti-chain of* $\mathcal{P}$.

An $X \in Y \subseteq \wp(A)$ is said to be *[K-]meet-irreducible in* Y, [where $K \subseteq \infty$], if $\forall Z \in \wp_{[K]}(Y) : ((A \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$, their set being denoted by $\mathrm{MI}^{[K]}(Y)$, "finitely-" standing for "ω-" within any related context. Next, a $\mathcal{U} \subseteq \wp(A)$ is said to be *upward-directed*, if $\forall \mathcal{S} \in \wp_\omega(\mathcal{U}) : \exists T \in (\mathcal{U} \cap \wp(\bigcup \mathcal{S}, A))$, subsets of $\wp(A)$ closed under unions of upward directed subsets being called *inductive*. Further, a *[finitary] closure operator over* A is any unary operation on $\wp(A)$ such that $\forall X \in \wp(A), \forall Y \in \wp(X) : (X \cup C(C(X)) \cup C(Y)) \subseteq C(X)[= (\bigcup C[\wp_\omega(X)])]$. Finally, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ containing A and closed under intersections of subsets containing A, any $\mathcal{B} \subseteq \mathcal{C}$ with $\mathcal{C} = \{A \cap (\bigcap \mathcal{S}) \mid \mathcal{S} \subseteq \mathcal{B}\}$ being called a *(closure) basis of* $\mathcal{C}$ and determining the closure operator $C_{\mathcal{B}} \triangleq \{\langle Z, A \cap (\bigcap(\mathcal{X} \cap \wp(Z, A)))\rangle \mid Z \in \wp(A)\}$ over A with $(\mathrm{img}\, C_{\mathcal{B}}) = \mathcal{C}$. Conversely, $\mathrm{img}\, C$ is a closure system over A with $C_{\mathrm{img}\, C} = C$, being inductive iff C is finitary, and forming a complete lattice under the partial ordering by inclusion with meet/join $(\Delta_{\wp(A)}/C)(A \cap ((\bigcap / \bigcup)\mathcal{S}))$ of any $\mathcal{S} \subseteq (\mathrm{img}\, C)$, C and $\mathrm{img}\, C$ being called *dual to* one another. Then, $C(X) \in (\mathrm{img}\, C)$ is said to be *generated by* an $X \subseteq A$, elements of $C[\wp_{\omega/\{n\}}(A)]$ /"with $n \in (\omega|\{1\})$" being said to be *finitely/n-generated|principal*.

Remark 2.1. Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element, $MI^{[K]}(\mathcal{C})$ is a basis of any inductive closure system $\mathcal{C}$. □

A *filter/ideal on* A is any $\mathcal{F} \subseteq \wp(A)$ such that, for all $\mathcal{S} \in \wp_\omega(\wp(A))$, $(\mathcal{S} \subseteq \mathcal{F}) \Leftrightarrow ((A \cap ((\bigcap / \bigcup)\mathcal{S})) \in \mathcal{F})$ "the set $\mathrm{Fi}(A)$ of them being an inductive closure system over $\wp(A)$ with dual finitary closure operator (of filter generation) Fg_A such that

$$\mathrm{Fg}_A(\mathcal{T}) = \wp(A \cap (\bigcap \mathcal{T}), A), \tag{2.3}$$

for all $\mathcal{T} \in \wp_\omega(\wp(A))$"/. Then, an *ultra-filter on* A is any filter $\mathcal{U}$ on A such that $\wp(A) \setminus \mathcal{U}$ is an ideal on A.

2.2. **Algebraic background.** Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature Σ, Σ-algebras/"their carriers" being denoted by same capital Fraktur/Italic letters (with same indices|suffixes|prefixes, if any) "with denoting the class of all [one-element] ones by $\mathsf{A}_\Sigma^{[=1]}$"/. In general, any notation, being specified for single Σ-algebras, is tacitly supposed to be extended to their classes member-wise. Given any $\alpha \in (\infty \setminus 1)$, let $\mathrm{Tm}_\Sigma^\alpha$ be the carrier of the absolutely-free Σ-algebra $\mathfrak{Tm}_\Sigma^\alpha$, freely-generated by the set $V_\alpha \triangleq \{x_\beta\}_{\beta\in\alpha}$ of *(first*

α*) variables*, and $\mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Tm}^\alpha_\Sigma)^2$, $\phi \approx /(\lessapprox \mid \gtrapprox)\psi$, where $\phi, \psi \in \mathrm{Tm}^\alpha_\Sigma$ /"and $\wedge \in \Sigma$", meaning $\langle \phi/(\phi \wedge \psi), \psi/(\phi|\psi)\rangle$ "and being called a Σ-*equation of rank* α"/. /"Likewise, for any Σ-algebra $\mathfrak{A}$ and $a, b \in A$, $(a(\leqslant \mid \geqslant)^{\mathfrak{A}}b)\|[a,b]_{\mathfrak{A}}$ stands for $((a|b) = (a \wedge^{\mathfrak{A}} b))\|\{c \in A \mid a \leqslant^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} b\}$." Then, any $\langle \Gamma, \Phi\rangle \in (\wp_{\infty/(1[\cup\omega])}(\mathrm{Eq}^\alpha_\Sigma) \times \mathrm{Eq}^\alpha_\Sigma)$ /"with $\alpha \in \omega$" is called a Σ-*implication/-[quasi-]identity of rank* α, written as $\Gamma \to \Phi$ and identified with Φ, if $\Gamma = \varnothing$, as well as treated as the universal infinitary/first-order strict Horn sentence $\forall_{\beta\in\alpha} x_\beta((\bigwedge \Gamma) \to \Phi)$, the class/set of those of any /finite rank true in a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ being called the *implicational/[quasi-]equational theory of* K and denoted by $(\mathcal{I}/[\mathcal{Q}]\mathcal{E})(\mathsf{K})$.

Subclasses of $\mathsf{A}_\Sigma\{\cap\mathsf{K}$ with $\mathsf{K} \subseteq \mathsf{A}_\Sigma\}$ "closed under $\{\mathsf{K}\cap\}(\mathbf{I}|\mathbf{H}|\mathbf{S}_{(>1)}|\mathbf{P}^{[\mathrm{SD}\|\mathrm{U}]})$"/ "containing each Σ-algebra with finitely-generated subalgebras in them"/"containing no infinite finitely-generated member" are called *"*$\{$*relatively*$\}$ *abstract|image-closed|(non-trivially-)hereditary|[ultra-\|sub-]multiplicative"/local/locally-finite* (cf. [15]). Then, a *skeleton* $\{$*of* a(n abstract) $\mathsf{K} \subseteq \mathsf{A}_\Sigma\}$ is any $\mathsf{S} \subseteq \mathsf{A}_\Sigma$ without pair-wise distinct isomorphic members $\{$such that $\mathsf{S} \subseteq \mathsf{K} \subseteq \mathbf{IS}$ (i.e., $\mathsf{K} = \mathbf{IS}$)$\}$. Given $\mathsf{K}, \mathsf{K}' \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$, set $\hom^{[\mathrm{S}]}_{(\mathrm{I})}(\mathfrak{A}, \mathsf{K}) \triangleq \{h \in \hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}[, (\mathrm{img}\, h) = B](, (\ker h) = \Delta_A)\}$ and $\mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, whose elements are called K-*(relative)congruences of* $\mathfrak{A}$, $\mathsf{K}' \preceq \mathsf{K}$ standing for $\mathsf{K}' \subseteq \mathbf{ISK}$ and thus providing a quasi-ordering on A_Σ, in which case, by the Homomorphism Theorem, we have

$$(\ker[\hom^{\mathrm{S}|}(\mathfrak{A}, \mathsf{K})]\{\setminus(\varnothing|\{A^2\})\}) = \mathrm{Co}_{(\mathbf{I}|(\mathbf{IS}_{\{>1\}}))\mathsf{K}}(\mathfrak{A}), \tag{2.4}$$

and so "by the Homomorphism Theorem"|, for all $\mathfrak{B} \in \mathsf{A}_\Sigma$ and $h \in \hom^{\mathrm{S}|(\mathrm{S}\|)}(\mathfrak{B}|\mathfrak{A}, \mathfrak{A}|\mathfrak{B})$:

$$\begin{aligned}\forall\theta \in (\mathrm{Co}_{[(\mathbf{I}|(\mathbf{I}\|(\mathbf{IS})))\mathsf{K}]}(\mathfrak{B}) \cap \wp((\ker h)|\Delta_B, B^2)) :\\ h^{|(-1)}_*[\theta] \in (\mathrm{Co}_{[(\mathbf{I}|(\mathbf{I}\|(\mathbf{IS})))\mathsf{K}]}(\mathfrak{A}) \cap \wp(\Delta_A|(\ker h), A^2)),\\ h^{(-1)|}_*[h^{|(-1)}_*[\theta]] = (\theta \cap (B|h[A])^2)\end{aligned} \tag{2.5}$$

"yielding an isomorphism between the posets $\mathrm{Co}_{[\mathbf{IK}]}(\mathfrak{B}) \cap \wp(\ker h, B^2)$ and $\mathrm{Co}_{[\mathbf{IK}]}(\mathfrak{A})$ ordered by inclusion as well as"$\| \mid$"implying:

$$h^{-1}_*[\mathrm{Cg}^{\mathfrak{B}}_{[(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}]}(h_*[X]) = \| \supseteq \mathrm{Cg}^{\mathfrak{A}}_{[(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}]}(X \cup (\ker h)), \tag{2.6}$$

for all $X \subseteq A^2$", while, as, for any set I, $\overline{\mathfrak{B}} \in \mathsf{A}^I_\Sigma$ and $\bar{f} \in (\prod_{i\in I} \hom(\mathfrak{A}, \mathfrak{B}_i))$:

$$(\prod\nolimits^{\mathrm{F}} \bar{f}) \in \hom(\mathfrak{A}, \prod_{i\in I} \mathfrak{B}_i), \tag{2.7}$$

by (2.1) and (2.2) with [finite] $I \triangleq \mathrm{Co}_{(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A})$ [if either A is finite or, by (2.4), both $\mathfrak{A}$ is finitely-generated and K as well as all its members are finite] for B, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{B}/i\rangle_{i\in I}$, $D \triangleq (\bigcup_{i\in I} B_i)$ and $\bar{f} \triangleq \langle \nu_i\rangle_{i\in I}$, we get:

$$(\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}_{[\omega]}(\{\mathbf{I}\}\|(\{\mathbf{I}\}\mathbf{S}))\mathsf{K})) \Leftrightarrow ((A^2 \cap (\bigcap \ker[\hom^{\mathrm{S}\|}(\mathfrak{A}, \mathsf{K})])) = \Delta_A), \tag{2.8}$$

whereas, since, for any $I \triangleq \Theta \subseteq \mathrm{Co}_{\langle\mathsf{K}\rangle}(\mathfrak{A})$, $\theta \triangleq (A^2 \cap (\bigcap \Theta)) \in \mathrm{Co}(\mathfrak{A})$, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{A}/i\rangle_{i\in I} \in (\mathsf{A}_{\boldsymbol{\Sigma}}\langle\cap\mathsf{K}\rangle)^I$ as well as, by the Homomorphism Theorem, $\bar{f} \triangleq \langle \nu^{-1}_\theta \circ \nu_i\rangle_{i\in I} \in (\prod_{i\in I} \hom(\mathfrak{A}/\theta, \mathfrak{B}_i))$, taking (2.1), (2.2) and (2.7) into account, we see that $e \triangleq (\prod \bar{f})$ is an embedding of $\mathfrak{A}/\theta$ into $\mathfrak{C} \triangleq (\prod_{i\in I} \mathfrak{B}_i)$ such that $\mathfrak{C}\restriction(\mathrm{img}\, e)$, being isomorphic to $\mathfrak{A}/\theta$, is a subdirect product of $\overline{\mathfrak{B}}$ $\langle$ in which case $(\mathfrak{A}/\theta) \in \mathbf{IP}^{\mathrm{SD}}\mathsf{K}$, and so, providing K is both abstract and sub-multiplicative, $\theta \in \mathrm{Co}_{\mathsf{K}}(\mathfrak{A})\rangle$. In particular, [providing K is both abstract and sub-multiplicative], $\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A})$ is a closure system over A^2, the dual closure operator being denoted by $\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{K}]}$.

Remark 2.2. By (2.4), the |-right alternative of (2.5) with $h = \nu_\vartheta$, where $\vartheta \in \mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}([\mathbf{I}]\|([\mathbf{I}]\mathbf{S}))\mathsf{K}}(\mathfrak{A})$, $\mathfrak{B} = (\mathfrak{A}/\vartheta)$ and $\theta = \Delta_B$ as well as (2.8), since $\vartheta = h_*^{-1}[\theta]$, while h_*^{-1} preserves intersections, $\mathrm{Co}_{(\mathbf{I}\|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A})$ is a basis of the closure system $\mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}([\mathbf{I}]\|([\mathbf{I}]\mathbf{S}))\mathsf{K}}(\mathfrak{A})$ over A^2. □

Given any Σ-algebra $\mathfrak{A}$ and any function f with $(\mathrm{dom}\, f) = A$ and $(\ker f) \in (\mathrm{Co}(\mathfrak{A})/\{\Delta_A\})$, we have its *homomorphic/isomorphic image/copy* $f[\mathfrak{A}]$ *by* f with carrier $f[A]$ and operations $\varsigma^{f[\mathfrak{A}]} \triangleq f_*[\varsigma^{\mathfrak{A}}]$, for each $\varsigma \in \Sigma$, in which case $f \in \hom^{\mathrm{S}}(\mathfrak{A}, f[\mathfrak{A}])$, and so $f[\mathfrak{A}] \in (\mathbf{H}|\mathbf{I})\mathfrak{A}$, such exhausting all members of $(\mathbf{H}|\mathbf{I})\mathfrak{A}$.

According to [23], *pre-varieties* are abstract hereditary multiplicative subclasses of A_Σ (these are exactly model classes of theories constituted by Σ-implications of unlimited rank, and so are also called *implicative/implicational*; cf., e.g., [4]/[18]), $\mathbf{PV}(\mathsf{K}) \triangleq \mathbf{ISP}\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}(\mathbf{I})\mathbf{S}_{[>1]}\mathsf{K} = \mathrm{Mod}(\mathfrak{I}(\mathsf{K}))$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Likewise, *[quasi-]varieties* are [ultra-multiplicative] pre-varieties closed under $\mathbf{H}^{[\mathrm{I}]}[\triangleq \mathbf{I}]$ (these are exactly model classes of sets of Σ-[quasi-]identities of unlimited finite rank, and so are local and also called *[quasi-]equational*; cf., e.g., [15]), $[\mathbf{Q}]\mathbf{V}(\mathsf{K}) \triangleq \mathbf{H}^{[\mathrm{I}]}\mathbf{SP}[\mathbf{P}^{\mathrm{U}}]\mathsf{K} = \mathrm{Mod}([\mathfrak{Q}]\mathfrak{I}(\mathsf{K}))$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Then, ((pre-/quasi-)varieties generated by finite classes of finite Σ-algebras are called *finitely-generated*, in which case, by [(2.8)] (and [6, Corollary 2.3]), they are locally-finite (and quasi-equational)/. Further, intersections of a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ with [pre-/quasi-]varieties are called its *relative sub-[pre-/quasi-]varieties*, in which case, for any $\mathcal{E} \subseteq \mathrm{Eq}^{\omega}_{\Sigma}$,

$$(\mathbf{IP}^{\mathrm{SD}}(\mathsf{K}) \cap \mathrm{Mod}(\mathcal{E})) = \mathbf{IP}^{\mathrm{SD}}(\mathsf{K} \cap \mathrm{Mod}(\mathcal{E})), \tag{2.9}$$

and so $\mathsf{S} \mapsto (\mathsf{S} \cap \mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the lattices of relative sub-varieties of $\mathbf{IP}^{\mathrm{SD}}\mathsf{K}$ and those of K.

Then, a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is said to be *[(relatively)] congruence-distributive*, if, for each $\mathfrak{A} \in \mathsf{P}$, $\mathrm{Co}_{[(\mathsf{P})]}(\mathfrak{A})$ is distributive.

Remark 2.3. Given a [quasi-equational] pre-variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ and $\alpha \in (\infty \setminus 1)$, by the |-right alternative of (2.4) with $\mathsf{K} = \mathsf{P}$ and $\mathfrak{A} = \mathrm{Tm}^{\alpha}_{\Sigma}$, any Σ-implication $\Gamma \to \Phi$ of rank α is true in P iff $\Phi \in \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\Gamma)$ [in which case, by the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [15]), $\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}$ is finitary, and so is $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{P}}$, for any $\mathfrak{B} \in \mathsf{A}_\Sigma$, in view of the left ‖-alternative of (2.6), when taking $\alpha = |B|$ and h to extend any bijection from V_α onto B]. □

Furthermore, [given an abstract $\mathsf{K} \subseteq \mathsf{A}_\Sigma$] an $\mathfrak{A} \in (\mathsf{A}_\Sigma[\cap\mathsf{K}])$ is said to be *[*K-{*relatively* }*]simple/(*K*-)subdirectly-irreducible* /(where $K \subseteq \infty$), if $\Delta_A \in (\max_{\subseteq} / \mathrm{MI}^{(K)})(\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, in which case $|A| \neq 1$, the class of ⟨those of⟩ them ⟨which are in a $\mathsf{K}' \subseteq (\mathsf{A}_\Sigma[\cap\mathsf{K}])$⟩ being denoted by $(\mathrm{Si} \,/\, \mathrm{SI}^{(K)})_{[\mathsf{K}]}\langle(\mathsf{K}')\rangle$,[1] and so, by (2.4) and (2.8),

$$(\mathrm{Si} \,|\, \mathrm{SI})_{[\mathbf{IP}^{\mathrm{SD}}(\mathbf{S})\mathsf{K}'']}(\mathbf{IP}^{\mathrm{SD}}(\mathbf{S})\mathsf{K}'') \subseteq \mathbf{I}(\mathbf{S}_{>1})\mathsf{K}'', \tag{2.10}$$

for any $\mathsf{K}'' \subseteq \mathsf{A}_\Sigma$. Then, a [pre-]variety P is said to be *[{relatively}] (finitely) semi-simple/subdirectly-representable*, if

$$(\mathrm{SI}^{(\omega)}_{[\{\mathsf{P}\}]}(\mathsf{P})/\mathsf{P}) \subseteq | = (\mathrm{Si}_{[\{\mathsf{P}\}]}(\mathsf{P})/\mathbf{IP}^{\mathrm{SD}}(\mathrm{Si} \,/\, \mathrm{SI}^{(\omega)})_{[\{\mathsf{P}\}]}(\mathsf{P})),$$

any variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ being well-known, due to Birkgoff's Theorem, to be subdirectly-representable. More generally, we have:

Remark 2.4. Given any [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ and $\mathfrak{A} \in (\{\mathsf{Q}\cap\}\mathsf{A}_\Sigma)$, by Remarks 2.1, 2.2, 2.3 and the right ‖-alternative of (2.5), $\mathrm{MI}^{(\omega)}(\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A})) = \mathrm{Co}_{\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})}(\mathfrak{A})$ is

[1] This is abstract ⟨whenever K' is so⟩, in view of (2.5).

a basis of both $\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A})$ and $\mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}\,\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})}(\mathfrak{A})$, in which case these are equal {and so, since $\nu_{\Delta_A} \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{A}/\Delta_A)$ is injective, $\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}\,\mathrm{SI}^{(\omega)}_{\mathsf{Q}}(\mathsf{Q})$}. In particular, Q is [relatively] (finitely) subdirectly-representable. □

Recall that, according to [14], a[n implicational] $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is *congruence-permutable*, i.e., for each $\mathfrak{A} \in \mathsf{K}$ and all $\theta, \vartheta \in \mathrm{Co}(\mathfrak{A})$, $(\theta \circ \vartheta) \subseteq (\vartheta \circ \theta)$, if[f] it has a *congruence-permutation term*, viz., a $\pi \in \mathrm{Tm}^3_\Sigma$ such that K satisfies the Σ-identities in $\{x_1 \approx (\sigma_i(\pi)) \mid i \in \{0,2\}\}$, where, for every $j \in 3$, $\sigma_j \triangleq [x_j/x_1; x_k/x_0]_{k\in(3\setminus\{j\})}$. Likewise, a *minority|majority term for* K |{with $\Sigma_+ \triangleq \{\wedge, \vee\} \subseteq \Sigma$ and the Σ_+-reducts of members of K being lattices} is any $\mu \in \mathrm{Tm}^3_\Sigma$ such that K satisfies the Σ-identities in $\{x_{(1-\min(2-i,i))|0} \approx (\sigma_i(\mu)) \mid i \in 3\}$ |{$\mu_+ \triangleq (\wedge_+ \langle x_i \vee (x_{\max(1-i,0)} \wedge x_{2+\min(i,1-i)})\rangle_{i\in 3})$ being so}, in which case it is so "as well as a congruence-permutation term"| for the variety generated by K, and so this is congruence-distributive [17], while, for any congruence-permutation term π for K, $\pi[x_1/\mu]$ is a majority|minority term for K "and so $\mu[x_1/\mu]$ is a majority term for K"|. Finally, a *(ternary) |dual discriminator (term) for* K is any $\delta \in \mathrm{Tm}^3_\Sigma$ such that, for each $\mathfrak{A} \in \mathsf{K}$, $\delta^{\mathfrak{A}} = ((\pi_{2|0}\restriction(\Delta_A \times A)) \cup (\pi_{0|2}\restriction((A^2 \setminus \Delta_A) \times A)))$, in which case $\mathfrak{A}$ is simple, because, for every $\theta \in (\mathrm{Co}(\mathfrak{A})\setminus\{\Delta_A\})$, any $\langle a, b\rangle \in (\theta\setminus\Delta_A) \neq \varnothing$ and all $c \in A$, we have $(a|c) = \delta^{\mathfrak{A}}(a, b, c)\ \theta\ \delta^{\mathfrak{A}}(a, a, c) = (c|a)$, so getting $\theta = A^2$, while δ is a |dual discriminator for $\mathbf{ISP}^{\mathrm{U}}\mathsf{K}$ as well as a minority|majority term for K, whereas, for any congruence-permutation term π for K, $\pi[x_1/\delta]$ is a dual| discriminator for K "and so $\delta[x_1/\delta]$ is a dual discriminator for K"|, {⟨quasi-/pre-⟩varieties generated by classes of} Σ-algebras with [dual] discriminator δ being called *[dual] δ-discriminator*, with denoting the class of [dual] δ-discriminator members of a $\mathsf{C} \subseteq \mathsf{A}_\Sigma$ by $\mathsf{C}^{[\partial]}_\delta$. Then, [dual] δ-discriminator quasi-varieties are exactly quasi-equational [dual] δ-discriminator pre-varieties.

2.2.1. *Filtral congruences.* Let I be a set, $\mathcal{F}$ a{n ultra-}filter on I [$\mathsf{P} \subseteq \mathsf{A}_\Sigma$ a (quasi-equational) pre-variety], $\overline{\mathfrak{A}} \in (\mathsf{A}_\Sigma[\cap\mathsf{P}])^I$ and $\mathfrak{B}$ a subalgebra of its direct product. Then, by (2.5), for each $i \in I$, $(B^2 \cap (\ker \pi_i)) = ((\pi_i\restriction B)^{-1}_2[\Delta_{A_i}] \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, as $(\pi_i\restriction B) \in \hom(\mathfrak{B}, \mathfrak{A}_i)$ and $\Delta_{A_i} \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i)$, in which case, for all $K \subseteq J \subseteq I$, the closure system $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$ on B^2 contains $\theta^B_J \triangleq (B^2 \cap \varepsilon^{-1}_I[\wp(J, I)]) = (B^2 \cap (\bigcap_{j\in J} \ker \pi_j)) \subseteq \theta^B_K$, $\Theta^B_{\mathcal{F}} \triangleq \{\theta^B_L \mid L \in \mathcal{F}\}$ being thus upward-directed (and so $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, being inductive, in view of Remark 2.3, contains $\theta^B_{\mathcal{F}} \triangleq (\bigcup \Theta^B_{\mathcal{F}}) = (B^2 \cap \varepsilon^{-1}_I[\mathcal{F}])$, called ⟨$\mathcal{F}$-⟩{*ultra-*}*filtral*). Clearly, for any $\mathfrak{X} \subseteq \mathrm{Fi}(I)$ |"with $(\bigcup \mathfrak{X}) \in \mathrm{Fi}(I)$",

$$\theta^B_{\wp(I)\cap((\bigcap\mid\bigcup)\mathfrak{X})} = (B^2 \cap ((\bigcap\mid\bigcup)\{\theta^B_{\mathcal{F}} \mid \mathcal{F} \in \mathfrak{X}\})). \tag{2.11}$$

A [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is said to be *[relatively] (subdirectly) ⟨finitely/principally⟩ filtral*, if every ⟨finitely-generated/principal⟩ [P-]congruence of each member of $\mathbf{SP}\,\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})(\cap\mathbf{P}^{\mathrm{SD}}\,\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}))$ is filtral (cf. [7] for the equational case).

2.2.1.1. Filtrality versus semi-simplicity.

Lemma 2.5. *Any [relatively] subdirectly principally filtral [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is [relatively] semi-simple.*

Proof. Consider any $\mathfrak{A} \in \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})$, in which case $|A| > 1$, and any $\theta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$ as well as any $\bar{a} \in (\theta\setminus\Delta_A) \neq \varnothing$, in which case $\mathfrak{B} \triangleq \mathfrak{A}^1 \in \mathbf{P}^{\mathrm{SD}}\,\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})$, while $h \triangleq (\pi_0\restriction B) \in \hom^{\mathrm{S}}(\mathfrak{B}, \mathfrak{A})$ is injective, whereas $B^2 \ni \bar{b} \triangleq (\bar{a} \circ h^{-1}) \in \vartheta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\bar{b}) = \theta^B_{\mathcal{F}}$, for some $\mathcal{F} \in \mathrm{Fi}(1)$, and so, by (2.5), $\eta \triangleq h^{-1}_*[\theta] \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \cap \wp(\vartheta, B^2))$, while $\theta = h_*[\eta]$, whereas $\varnothing = \varepsilon_1(\bar{b}) \in \mathcal{F}$. Then, $\mathcal{F} = \wp(1)$, in which case $\eta \supseteq \vartheta = B^2$, and so $\theta \supseteq h_*[B^2] = A^2$. Thus, $\mathfrak{A} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, as required. □

2.2.1.2. Filtrality versus congruence-distributivity.

Lemma 2.6 (cf. [10] for the []()-non-optional case)**.** *Let* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *be a [quasi-]variety, I a set,* $\overline{\mathfrak{A}} \in \mathsf{Q}^I$, $\mathfrak{B} \in \mathbf{S}(\prod \overline{\mathfrak{A}})$ *and* $\theta \in \mathrm{MI}^{(\omega)}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$. *Suppose* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *is distributive. Then, there is an ultra-filter* $\mathcal{U}$ *on I such that* $\theta^B_{\mathcal{U}} \subseteq \theta$.

Proof. By (2.11), $S \triangleq \{\mathcal{F} \in \mathrm{Fi}(I) \mid \theta^B_{\mathcal{F}} \subseteq \theta\} \ni \{I\}$ is inductive, for $\mathrm{Fi}(I)$ is so, in which case, by Zorn Lemma, it, being non-empty, has a maximal element $\mathcal{U}$, and so, for any $\mathcal{X} \in \wp_\omega(\wp(I))$ such that $Y \triangleq (\bigcup \mathcal{X}) \in \mathcal{U}$, $(\mathcal{X} \cap \mathcal{U}) \neq \varnothing$, as, for each $Z \in \mathcal{X}$, $\theta^B_{\mathcal{F}_Z} \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ with $\mathcal{U} \subseteq \mathcal{F}_Z \triangleq \mathrm{Fg}_I(\mathcal{U} \cup \{Z\}) \in \mathrm{Fi}(I)$, while $\mathcal{U} = \mathrm{Fg}_I(\mathcal{U}) = \mathrm{Fg}_I(\mathcal{U} \cup \{Y\}) = (\wp(I) \cap (\bigcap\{\mathcal{F}_Z \mid Z \in \mathcal{X}\}))$, in view of (2.3), since Fg_I is finitary, whereas, by (2.11), $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{U}}) = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup (B^2 \cap (\bigcap\{\theta^B_{\mathcal{F}_Z} \mid Z \in \mathcal{X}\}))) = (B^2 \cap (\bigcap\{\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \mid Z \in \mathcal{X}\}))$, that is, for some $Z \in \mathcal{X}$, $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \supseteq \theta^B_{\mathcal{F}_Z}$, i.e., $\mathcal{U} \subseteq \mathcal{F}_Z \in S$, viz., $Z \in \mathcal{F}_Z = \mathcal{U}$, as required. □

This, by (2.5), Birkgoff's and the Homomorphism Theorems [as well as [6, Corollary 2.3]/[21, Lemma 2.1]], immediately yields:

Corollary 2.7. *Let* K *be a [finite/] class of [finite/]* Σ*-algebras (with* {*dual*} *discriminator* δ*) and* $\mathsf{P} \triangleq \mathbf{H}^{\langle \mathrm{I} \rangle}\mathbf{SP}\mathsf{K}$. *Suppose* P *is a* ⌈*relatively*⌉ *congruence-distributive [/locally-finite]* ⌈*quasi-*⌉*variety. Then,*

$$(\mathsf{P}^{\{\partial\}}_\delta \subseteq \mathrm{Si}_{\lfloor \lceil \mathsf{P} \rceil \rfloor}(\mathsf{P}) \subseteq) \, \mathrm{SI}^{\omega|\infty}_{\lceil \mathsf{P} \rceil}(\mathsf{P}) \subseteq \mathbf{H}^{(\|\langle \mathrm{I} \rangle \|)}\mathbf{SP}^{\mathrm{U}}\mathsf{K}[\subseteq \mathbf{H}^{(\|\langle \mathrm{I} \rangle \|)}\mathbf{S}\mathsf{K}](\subseteq \mathsf{P}^{\{\partial\}}_\delta)$$

[in which case its members are finite, and so $\mathrm{SI}^{\omega}_{\lceil \mathsf{P} \rceil}(\mathsf{P}) = \mathrm{SI}_{\lceil \mathsf{P} \rceil}(\mathsf{P})$*]/. In particular,* {*dual*} *(*δ*-)discriminator quasi-varieties are exactly [semi-simple]* {*dual*} *(*δ*-)discriminator varieties.*

Corollary 2.8. *Let* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *be a ([relatively] semi-simple) [quasi-]variety,* $I \in \Upsilon$, $\overline{\mathfrak{A}} \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$, $\mathfrak{D} \triangleq (\prod \overline{\mathfrak{A}})$, $\mathfrak{B} \in \mathbf{S}\{\mathfrak{D}\}$ *and* $\theta \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. *Suppose* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$ *is both ultra-multiplicative and non-trivially-hereditary* {*while* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *is distributive*}*. Then,* θ *is maximal in* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}$ *if*{*f*} *it is ultra-filtral.* {*(In particular, all elements of* $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ *are filtral.)*}

Proof. First, assume $\theta = \theta^B_{\mathcal{U}}$, for some ultra-filter $\mathcal{U}$ on I, in which case $\mathfrak{C} \triangleq (\mathfrak{D}/\theta^D_{\mathcal{U}}) \in \mathbf{P}^{\mathrm{U}}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$, while $h \triangleq (\Delta_B \circ \nu_{\theta^D_{\mathcal{U}}}) \in \hom(\mathfrak{B}, \mathfrak{C})$, whereas $(\ker h) = (\Delta_B)^{-1}_*[\theta^D_{\mathcal{U}}] = \theta$, and so by (2.4) and Footnote 1, as $\theta \neq B^2$, $(\mathfrak{B}/\theta) \in \mathbf{IS}_{>1}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$. Then, by (2.5), $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. {Conversely, assume $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \subseteq \mathrm{MI}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$, in which case, by Lemma 2.6, there is some ultra-filter $\mathcal{U}$ on I such that, as $\theta \neq B^2$, $(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \ni \theta^B_{\mathcal{U}} \subseteq \theta$, and so, by the "if" part, $\theta = \theta^B_{\mathcal{U}}$. (Then, Remarks 2.1, 2.3, 2.4, (2.5) and (2.11) complete the argument.)} □

2.2.2. *Subdirect products versus subalgebras.*

Lemma 2.9 (cf. [12])**.** *Let* $\mathfrak{A} \in \mathsf{A}_\Sigma$ *and* $\mathfrak{B}$ *a subalgebra of* $\mathfrak{A}$. *Then,* $h^B_A \triangleq \{\langle \bar{a}, b\rangle \in (A^\omega \times B) \mid |\omega \setminus \varepsilon_\omega(\bar{a}, \omega \times \{b\})| \in \omega\} \supseteq (\bigcup\{\{\langle \omega \times \{b\}, b\rangle\} \cup \{\langle((\omega \setminus \{i\}) \times \{b\}) \cup \{\langle i, a\rangle\}, b\rangle \mid i \in \omega, a \in A\} \mid b \in B\})$ *is a function forming a subalgebra of* $\mathfrak{A}^\omega \times \mathfrak{B}$*, in which case it is a surjective homomorphism from* $\mathfrak{C}^B_A \triangleq (\mathfrak{A}^\omega \restriction (\mathrm{dom}\, h^B_A))$ *onto* $\mathfrak{B}$*, and so* $\mathfrak{C}^B_A$ *is a subdirect product of* $\omega \times \{\mathfrak{A}\}$. *In particular, the variety generated by any* $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ *is equal to* $\mathbf{IP}^{\mathrm{SD}}\mathsf{K}$.

2.2.2.1. Filtrality versus non-trivial hereditarity of simplicity.

Corollary 2.10. *Let* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *be a [relatively] subdirectly principally filtral [pre-]variety. Then,* $(\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)(\setminus \mathsf{A}^{=1}_\Sigma)$ *is (non-trivially-)hereditary.*

Proof. Let $\mathfrak{A} \in (\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1})$ and $\mathfrak{B}$ a non-one-element subalgebra of $\mathfrak{A}$, in which case $|A| \neq 1$, and so, by Lemma 2.9, $h \triangleq h_A^B$ is a surjective homomorphism from the subdirect product $\mathfrak{C} \triangleq \mathfrak{C}_A^B$ of $(\omega \times \{\mathfrak{A}\}) \in \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})^\omega$ onto $\mathfrak{B}$. Consider any $\theta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \setminus \{\Delta_B\})$ and take any $\langle a, b\rangle \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case, by (2.5), $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{C}) \ni \vartheta \triangleq h_*^{-1}[\theta] \ni \langle \bar{c}, \bar{d}\rangle \triangleq \langle \omega \times \{a\}, \omega \times \{b\}\rangle$, while $h_*[\vartheta] = \theta$, and so $\vartheta \supseteq \eta \triangleq \mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{P}]}(\langle \bar{c}, \bar{d}\rangle) = \theta^C_{\mathcal{F}}$, for some $\mathcal{F} \in \mathrm{Fi}(\omega)$. Then, $\varnothing = \varepsilon_\omega(\bar{c}, \bar{d}) \in \mathcal{F}$, in which case $\mathcal{F} = \wp(\omega)$, and so $\vartheta \supseteq \eta = C^2$. Thus, $\theta \supseteq h_*[C^2] = B^2$, in which case $\theta = B^2$, and so $\mathfrak{B} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, as required. □

2.2.3. *Locality versus local finiteness.* As an immediate consequence of [21, Lemma 2.1], in its turn, being that of [6, Corollary 2.3], we, first, have the following useful universal observation:

Corollary 2.11. *Any abstract hereditary local subclass of a locally-finite quasi-variety is ultra-multiplicative.*

Aside from quasi-varieties as such, certain representative subclasses of them are local as well.

2.2.3.1. Local subclasses of local pre-varieties.

Lemma 2.12. *Let $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ be a [local (more specifically, quasi-equational) pre-]variety. Then, $(\mathrm{SI}^\omega \,|\, \mathrm{Si})_{[\mathsf{P}]})(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1})$ is local.*

Proof. Consider any $\mathfrak{B} \in (\mathsf{P} \setminus ((\mathrm{SI}^\omega \,|\, \mathrm{Si})_{[\mathsf{P}]})(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1}))$, in which case there are some $\bar{a} \in (B^2 \setminus \Delta_B) \neq \varnothing$, $n \in (\omega|\{1\})$ and $\bar{\theta} \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B}) \setminus (\mathrm{img}\, \bar{\vartheta}^B))^n$, where, for any $C \subseteq B$, $\bar{\vartheta}^C \triangleq (\langle \Delta_C \rangle | \langle \Delta_C, C^2\rangle)$, "such that $(B^2 \cap (\bigcap(\mathrm{img}\, \bar{\theta}))) = \Delta_B$"|, and so some $\langle \bar{b}^{i,j}\rangle_{i\in n}^{j\in(1|2)} \in (\prod_{i\in n}^{j\in(1|2)}((\theta_i \setminus \vartheta_j^B) \cup (\vartheta_j^B \setminus \theta_i))) \neq \varnothing$. Let $\mathfrak{A}$ be the finitely-generated subalgebra of $\mathfrak{B}$ generated by $\{a_0, a_1\} \cup \{b_k^{i,j} \mid i \in n, j \in (1|2), k \in 2\}$, in which case, by (2.5) with $h = \Delta_A$, $\bar{\eta} \triangleq \langle \theta_i \cap A^2\rangle_{i\in n} \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus (\mathrm{img}\, \bar{\vartheta}^A))^n$, as $\langle \bar{b}^{i,j}\rangle_{i\in n}^{j\in(1|2)} \in (\prod_{i\in n}^{j\in(1|2)}((\eta_i \setminus \vartheta_j^A) \cup (\vartheta_j^A \setminus \eta_i)))$, so $\mathfrak{A} \in (\mathsf{P} \setminus ((\mathrm{SI}^\omega \,\mathrm{Si})_{[\mathsf{P}]})(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1}))$, for $\bar{a} \in (A^2 \setminus \Delta_A)$ "and $(A^2 \cap (\bigcap(\mathrm{img}\, \bar{\eta}))) = (A^2 \cap (\bigcap(\mathrm{img}\, \bar{\theta}))) = (A^2 \cap \Delta_B) = \Delta_A$"|. □

2.2.3.1.1. Finite semi-simplicity versus semi-simplicity and local finiteness. Lemma 2.12 immediately yields:

Corollary 2.13. *Any locally-finite [relatively] semi-simple [local (more specifically, quasi-equational) pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ with hereditary $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}_\Sigma^{=1}$ is [relatively] finitely semi-simple.*

2.2.4. *Subdirect irreducibility versus homomorphisms onto simple algebras.*

Lemma 2.14. *Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{A}_\Sigma$ and $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B})$ (as well as $n \in (\omega \setminus 2)$). (Suppose $\mathfrak{B}$ is simple, while $|A| = n$, whereas $|B| = (n-1)$.) Then, $\mathfrak{A}$ is subdirectly-irreducible if(f) its congruences form the three-element chain $\Delta_A \subsetneq (\ker h) \subsetneq A^2$.*

Proof. The "if" part is immediate. (Conversely, assume $\mathfrak{A}$ is subdirectly-irreducible, in which case $\theta \triangleq (A^2 \cap (\bigcap(\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\}))) \in (\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$, and so, by (2.5), $\{\theta \subseteq\}(\ker h) \in [\max_\supseteq](\mathrm{Co}(\mathfrak{A}) \setminus \{\{\Delta_A, \}A^2\})$, for $1 \neq n\{\not\leqslant (n-1)\}$. Then, by the Homomorphism Theorem, $g \triangleq (\nu_\theta^{-1} \circ h) \in \hom^{\mathrm{S}}(\mathfrak{A}/\theta, \mathfrak{B})$, in which case $(n-1) \leqslant |A/\theta| < n$, for $\theta \neq \Delta_A$, and so $|A/\theta| = (n-1)$. Thus, g is injective, in which case $(\ker h) = \theta$, and so its maximality completes the argument.) □

3. Preliminaries: quaternary equational schemes

A *quaternary Σ-(equational)scheme* is any $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$. This is called an *implication scheme for* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if this satisfies the Σ-implication:

$$(\{x_0 \approx x_1\} \cup \mathfrak{V}) \to (x_2 \approx x_3). \tag{3.1}$$

Likewise, it is called an *identity|reflexive|symmetric|transitive* one, if K satisfies the Σ-implications of the form $(\varnothing|\varnothing|\mathfrak{V}|(\mathfrak{V}\cup(\mathfrak{V}[x_{2+i}/x_{3+i}]_{i\in 2})))\to\Psi$, where $\Psi\in(\mathfrak{V}([x_3/x_2]|[x_{2+i}/x_i]_{i\in 2}|[x_3/x_2,x_2/x_3]|[x_3/x_4]))$, reflexive symmetric transitive ones being also called *equivalence* ones. Then, $\mathfrak{V}$ is called a *congruence* one, if it is an equivalence one, while, for each $\varsigma\in\Sigma$ of arity $n\in(\omega\setminus 1)$, K satisfies the Σ-implications of the form $(\bigcup_{j\in n}(\mathfrak{V}[x_{2+i}/x_{2+i+(2\cdot j)}]_{i\in 2}))\to\Psi$, where $\Psi\in(\mathfrak{V}[x_{2+i}/\varsigma(\langle x_{2+i+(2\cdot j)}\rangle_{j\in n})]_{i\in 2})$.] Finally, $\mathfrak{V}$ [being finite] is called a *"restricted equationally definable principal {relative} congruence (REDP{R}C)"/"(equational) implicative|disjunctive scheme/system for* a "{pre-}variety"/ $\mathsf{K}\subseteq\mathsf{A}_\Sigma$, if, for each $\mathfrak{A}\in\mathsf{K}$ and all $\bar a\in A^4$, $(\forall\theta\in(\mathrm{Co}_{\{\mathsf{K}\}}(\mathfrak{A})/\{\Delta_A\}):(\langle a_0,a_1\rangle\in|\notin\theta)\Rightarrow(\langle a_3,a_3\rangle\in\theta))\Leftrightarrow(\mathfrak{A}\models(\bigwedge\mathfrak{V})[x_i/a_i]_{i\in I}$ [cf. [7]/[21]|[20]] /"and so for $\mathbf{IS}[\mathbf{P}^{\mathrm{U}}]\mathsf{K}$, ⟨pre-varieties generated by classes of⟩ Σ-algebras with [finite] implicative|disjunctive system $\mathfrak{V}$ being called ⟨*[finitely]*⟩ $\mathfrak{V}$*-implicative|-disjunctive* with the class of $\mathfrak{V}$-implicative|-disjunctive members of any $\mathsf{K}'\subseteq\mathsf{A}_\Sigma$ denoted by $\mathsf{K}'_{\mathfrak{V}}$ "in which case $\mathfrak{V}$, being an implication scheme for (the pre-variety generated by) K, providing this is quasi-equational, includes a finite one, by the Compactness Theorem for ultra-multiplicative classes of algebras [15]"|, and so implicative quasi-varieties, being thus finitely so, are exactly those in the original sense of [21]. Then, by Remark 2.4 therein, quasi-equational/finitely implicative pre-varieties are finitely disjunctive.

Given any $\mathfrak{A}\in\mathsf{A}_\Sigma$, let $\mathfrak{V}^{\mathfrak{A}}:\wp(A)^2\to\wp(A),\langle X,Y\rangle\mapsto\{\langle\phi_0^{\mathfrak{A}}[x_i/a_i,x_{2+i}/b_i]_{i\in 2},\phi_1^{\mathfrak{A}}[x_i/a_i,x_{2+i}/b_i]_{i\in 2}\rangle\mid\bar\phi\in\mathfrak{V},\bar a\in X,\bar b\in Y\}$.

Given any $\tau\in\mathrm{Tm}^3_\Sigma$, put

$$\begin{aligned}\mathfrak{V}^{\supset}_{\tau}&\triangleq\{\tau\approx(\tau[x_2/x_3])\},\\ \mathfrak{V}^{\partial\supset}_{\tau}&\triangleq\{(\tau[x_0/x_{2+k},x_1/x_{3-k},x_2/(\tau[x_2/x_{2+k}])])\approx x_{2+k}\mid k\in 2\},\\ \mathfrak{V}^{\vee}_{\tau}&\triangleq\{(\tau[x_0/\tau,x_1/(\tau[x_2/x_3])])\approx(\tau[x_0/\tau,x_1/(\tau[x_2/x_3]),x_2/x_3])\},\end{aligned}$$

in which case $\mathfrak{V}^{\vee}_{\tau}$ is defined by $\mathfrak{V}^{\supset}_{\tau}$ according to [21, Remark 2.4].

Remark 3.1. Given any [dual] discriminator $\tau\in\mathrm{Tm}^3_\Sigma$ for a $\mathsf{K}\subseteq\mathsf{A}_\Sigma$, $\mathfrak{V}_\tau^{([\partial]\supset)/\vee}$ is a finite implicative/disjunctive system for K. In particular, any [dual] discriminator pre-variety is finitely both implicative and disjunctive. □

This enables us to build easily an example of a non-quasi-equational finitely both implicative and disjunctive pre-variety well-justifying the generic framework of pre-varieties we follow here:

Example 3.2. Let $\Sigma=\{\neg,\nabla,\tau\}$, where $\neg$ and ∇ are unary, while τ is ternary, $\mathfrak{A}$ the Σ-algebra such that $A\triangleq\omega$, $\tau(x_0,x_1,x_2)$ is a (dual) discriminator for $\mathfrak{A}$ and, for all $a\in A$, $\nabla^{\mathfrak{A}}(a)\triangleq\min(a,1)$, whereas $\neg^{\mathfrak{A}}(a)\triangleq\max(0,a-1)$. Then, by Remark 3.1, the pre-variety P generated by $\mathfrak{A}$, being (dual) τ-discriminator, is finitely both implicative and disjunctive. Let us show, by contradiction, that it is not a quasi-variety. For suppose it is a quasi-variety. By induction on any $n\in\omega$, put $\neg^{0[+n+1]}x_i\triangleq[\neg\neg^n]x_i[=\neg^n\neg x_i]$, where $i\in 2$, and set $\varepsilon^n\triangleq(\nabla(\neg^n x_0)\approx\nabla(\neg^n x_1))$. Then, given any $N\subseteq\omega$, set $\varepsilon_N\triangleq\{\varepsilon^n|n\in N\}$. Note that the Σ-implication $\varepsilon_\omega\to(x_0\approx x_1)$ is true in $\mathfrak{A}$, and so in P. Hence, by Remark 2.3, there is some $N\in\wp_\omega(\omega)$ such that the Σ-quasi-identity $\varepsilon_N\to(x_0\approx x_1)$ is true in $\mathsf{P}\ni\mathfrak{A}$. However, $\mathfrak{A}\models\varepsilon_N[x_i/(i+m+1)]_{i\in 2}$, where $m\triangleq(\bigcup N)\in\omega$, though $(m+1)\neq(m+2)$. This contradiction means that P is not a quasi-variety. □

3.1. **Implicativity versus REDPRC and relative semi-simplicity.**

Lemma 3.3. *Let $\mathfrak{V}\subseteq\mathrm{Eq}^4_\Sigma$ be an implication scheme for a [pre-]variety $\mathsf{P}\subseteq\mathsf{A}_\Sigma$, $\mathfrak{A}\in\mathsf{P}$, $\bar a,\bar b\in A^2$ and $\theta\triangleq\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar a)$. Suppose $\mathfrak{A}\models(\bigwedge\mathfrak{V})[x_i/a_i,x_{2+i}/b_i]_{i\in 2}$. Then, $\bar b\in\theta$.*

Proof. As (3.1) is true in $\mathsf{P} \ni (\mathfrak{A}/\theta) \models (\bigwedge \mathfrak{V})[x_i/\nu_\theta(a_i), x_{2+i}/\nu_\theta(b_i)]_{i\in 2}$, while $\bar{a} \in \theta = (\ker \nu_\theta)$, we get $\bar{b} \in \theta$. □

Corollary 3.4. *Let $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$ be an implication/REDPC scheme for a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$. Then, $\mathsf{P}_\mathfrak{V} \subseteq \mathbb{I} = (\mathrm{Si}_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma)$. In particular, any implicative [pre-]variety is [relatively] both semi-simple and subdirectly representable.*

Proof. Consider any non-one-element $\mathfrak{A} \in \mathsf{P}_\mathfrak{V}$ and $\vartheta \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$, in which case there is some $\bar{a} \in (\vartheta \setminus \Delta_A) \neq \varnothing$, and so, for any $\bar{b} \in A^2$, $\mathfrak{A} \models (\bigwedge \mathfrak{V})[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$. Then, "by Lemma 3.3"/ $\bar{b} \in \vartheta$, in which case $\vartheta = A^2$, and so $\mathfrak{A}$ is [P-]simple. Conversely, for any $\mathsf{A} \in \mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})$, $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) = \{\Delta_A, A^2\}$, in which case, for all $\bar{a} \in A^4$, as $\langle a_2, a_3\rangle \in A^2$, we have $(\forall \theta \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) : (a_0\ \theta\ a_1) \Rightarrow (a_2\ \theta\ a_3)) \Leftrightarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$, and so $\mathfrak{A}$ is $\mathfrak{V}$-implicative, whenever $\mathfrak{V}$ is an REDP[R]C scheme for $\mathsf{P} \ni \mathfrak{A}$. □

Theorem 3.5. *Any $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$ is an identity congruence implication scheme for a[n equational] pre-variety $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ if[f] it is an REDPC one.*

Proof. The "if" part is immediate. [Conversely, if $\mathfrak{V}$ is an identity congruence implication scheme for K, then, by induction on construction of any $\varphi \in \mathrm{Tm}^\omega_\Sigma$, we conclude that K satisfies the Σ-identities in $\mathfrak{V}[x_{2+i}/(\varphi[x_0/x_i])]_{i\in 2}$, in which case, by Mal'cev Lemma [14] (cf. [7, Lemma 2.1]), for any $\mathfrak{A} \in \mathsf{A}$, $\bar{a} \in A^2$ and $\bar{b} \in \mathrm{Cg}^\mathfrak{A}(\bar{a})$, we have $\mathfrak{A} \models (\bigwedge \mathfrak{V})[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$, and so Lemma 3.3 completes the argument]. □

This, by Lemma 3.3 and the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [15]), immediately yields:

Corollary 3.6. *Any quasi-variety with REDPRC scheme $\mathfrak{V}$ has a finite one $\subseteq \mathfrak{V}$.*

Theorem 3.7. *Let $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$. Then, any [(not necessarily) quasi-equational pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is $\mathfrak{V}$-implicative iff it is [relatively (both subdirectly-representable and)] semi-simple with REDP[R]C scheme $\mathfrak{V}$, in which case $((\mathrm{SI} \,|\, \mathrm{Si})_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{=1}_\Sigma) = \mathsf{P}_\mathfrak{V}$.*

Proof. If P is $\mathfrak{V}$-implicative, that is, is the pre-variety generated by $\mathsf{P}_\mathfrak{V}$, then, for any $\mathfrak{A} \in \mathsf{P}$ and $\bar{a} \in A^4$ such that $\mathfrak{A} \not\models (\bigwedge \mathfrak{V})[x_i/a_i]_{i\in 4}$, by (2.8), there are some $\mathfrak{B} \in \mathsf{P}_\mathfrak{V}$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\mathfrak{B} \not\models (\bigwedge \mathfrak{V})[x_i/h(a_i)]_{i\in 4}$, that is, $h(a_{0|2}) = | \neq h(a_{1|3})$, in which case, by (2.4), $\langle a_{0|2}, a_{1|3}\rangle \in | \notin (\ker h) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A})$, and so Remark 2.4, Lemma 3.3 and Corollary 3.4 complete the argument. □

3.1.1. *REDPC versus congruence decomposition.*

Definition 3.8. Given any $n \in \omega$, a $\mathfrak{V} \subseteq \mathrm{Eq}^{2\cdot(n+1)}_\Sigma$ is called a(n) *restricted equationally definable n-generated [relative] congruence (n-REDG[R]C) scheme for* a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, if, for each $\mathfrak{A} \in \mathsf{P}$ and every $\bar{a} \in (A^2)^{n+1}$, $(a_n \in \mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\bar{a}[n])) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mathfrak{V})[x_{i+j}/\pi_j(a_i)]_{i\in(n+1), j\in 2})$. □

Given any $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$, by induction on any $n \in \omega$, define $\mathfrak{V}_n \subseteq \mathrm{Eq}^{2\cdot(n+1)}_\Sigma$ by $\mathfrak{V}_0 \triangleq \{x_0 \approx x_1\}$ and $\mathfrak{V}_{n+1} \triangleq (\bigcup\{\mathfrak{V}_n[x_{(2\cdot n)+i}/\varphi_i]_{i\in 2} \mid \bar{\varphi} \in (\mathfrak{V}[x_j/x_{(2\cdot n)+j}]_{j\in 4})\})$.

Lemma 3.9. *For any [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ with a REDP[R]C scheme $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$ and any $n \in \omega$, $\mathfrak{V}_n$ is an n-REDPG[R]C scheme for P.*

Proof. By induction on n. For consider any $\mathfrak{A} \in \mathsf{P}$, in which case Δ_A is the least [P-]congruence of $\mathfrak{A}$, and so $\mathfrak{V}_0$ is a 0-REDPGRC scheme for P. Now, assume $\mathfrak{V}_n$ is an n-REDPGRC scheme for P and consider any $\bar{a} \in (A^2)^{n+2}$, in which case, by the right alternative of (2.6) with $\mathfrak{B} = (\mathfrak{A}/\theta) \in \mathsf{P}$ and $h = \nu_\theta \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B})$, where $\theta \triangleq \mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\bar{a}[n]) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A})$, as $\theta = (\ker h)$, we have

$(a_{n+1} \in \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\bar{a}[n+1]) = \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{P}]}(\theta \cup \{a_n\})) \Leftrightarrow (h_*(a_{n+1}) \in \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\{h_*(a_n)\})) \Leftrightarrow (\mathfrak{B} \models (\bigwedge \mathfrak{V})[x_{(2\cdot i)+j}/h(\pi_j(a_{n+i}))]_{i,j\in 2}) \Leftrightarrow (\mathfrak{V}^{\mathfrak{A}}(\{a_n\},\{a_{n+1}\}) \subseteq \theta) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mathfrak{V}_{n+1})[x_{k+l}/\pi_l(a_k)]_{k\in(n+2),l\in 2})$, and so $\mathfrak{V}_{n+1}$ is an $(n+1)$-REDPGRC scheme for P. □

Theorem 3.10. *Let $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ be a ([relatively] semi-simple) {[quasi-]equational} [pre-]variety, $\mathfrak{V} \subseteq \mathrm{Eq}^4_\Sigma$, $I \in \Upsilon$, $\overline{\mathfrak{A}} \in (\mathrm{SI}_{[\mathsf{P}]})(\mathsf{P})^I$, $\mathfrak{B}$ a subalgebra of $\prod_{i\in I}\mathfrak{A}_i$ and $X \in \wp_{\omega|\infty}(B^2)$. Suppose $\mathfrak{V}$ is an REDP[R]C scheme for P, while $|B| \in (\infty|\omega)$. Then, $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X) = (B^2 \cap (\bigcap_{i\in I}(\pi_i\restriction B)^{-1}_*[\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i\restriction B)_*[X])]))(= \theta^B_{\wp(I\cap(\bigcap \varepsilon_I[X]),I)})$, in which case (all elements of)*

$$\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}[\wp_{\langle(\omega|\infty)\cup\rangle(\langle\varnothing\cap\rangle(\omega\{\cup\infty\}))}(B^2)] \quad \langle\subseteq \mid = \{B^2(\bigcap_{i\in I}(\pi_i\restriction B)^{-1}_*[\theta_i]) \mid \bar{\theta} \in (\prod_{i\in I}\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}_i))\}\rangle \tag{3.2}$$

(are filtral in the $\langle\rangle$-non-optional case {and so is P}). In particular, any /"finitely‖ [quasi-]equational" implicative [pre-]variety is [relatively] both subdirectly-representable and finitely/ filtral.

Proof. Take a bijection $\bar{a}$ from $n \triangleq |X| \in \omega$ to X, in which case, by Lemma 3.9, $\forall \bar{b} \in B^2 : (\bar{b} \in \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(X)) \Leftrightarrow (\mathfrak{B} \models (\bigwedge \mathfrak{V}_n)[x_{j+k}/\pi_k(a_j); x_{(2\cdot n)+l}/b_l]_{j\in n;k,l\in 2}) \Leftrightarrow (\forall i \in I : \mathfrak{A}_i \models (\bigwedge \mathfrak{V}_n)[x_{j+k}/\pi_i(\pi_k(a_j)); x_{(2\cdot n)+l}/\pi_i(b_l)]_{j\in n;k,l\in 2}) \Leftrightarrow (\forall i \in I : (\pi_i\restriction B)_*(\bar{b}) \in \mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i\restriction B)_*[X])) \Leftrightarrow (\bar{b} \in (B^2 \cap (\bigcap_{i\in I}(\pi_i\restriction B)^{-1}_*[\mathrm{Cg}^{\mathfrak{A}_i}_{[\mathsf{P}]}((\pi_i\restriction B)_*[X])])))(\Leftrightarrow ((I \cap (\bigcap \varepsilon_I[X])) \subseteq (\varepsilon_I(\bar{b}))) \Leftrightarrow (\bar{b} \in \theta^B_{\wp(I\cap(\bigcap \varepsilon_I[X]),I)}))$, and so $\langle$(3.2) holds $\mid$"in view of (2.5)", while$\rangle$ ({for every $\eta \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, since, by Remark 2.3, $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}$ is finitary, whereas $\mathrm{Fi}(I)$, being inductive, contains $\mathcal{F} \triangleq (\bigcup_{Y\in\wp_\omega(\eta)}\wp(I \cap (\bigcap \varepsilon_I[Y]), I))$, for $\{\wp(I \cap (\bigcap \varepsilon_I[Z]), I) \mid Z \in \wp_\omega(\eta)\} \subseteq \mathrm{Fi}(I)$ is upward-directed, as $\wp_\omega(\eta)$ is so, by (2.11), $\eta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\eta) = (\bigcup \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}[\wp_\omega(\eta)]) = \theta^B_{\mathcal{F}}$}. Finally, assume $\mathfrak{V}$ is finite, in which case, by Theorem 3.7, it, being an implicative system for $\mathsf{K}' \triangleq \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})$, is so for $\mathsf{K}'' \triangleq \mathbf{IP}^{\mathrm{U}}\mathsf{K}' \supseteq \mathsf{K}'$, and so $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K}') = \mathbf{PV}(\mathsf{K}'') \supseteq \mathbf{PV}(\mathsf{K}') = \mathsf{P}$, being $\mathfrak{V}$-implicative, is [relatively] semi-simple with REDPC scheme $\mathfrak{V}$ and $\mathrm{SI}_{[\mathsf{P}]}(\mathsf{P}) \supseteq \mathsf{K}'$. Then, by the {}-optional case, P is filtral, for Q is so.) In this way, Corollary 3.4 and Theorem 3.7 complete the argument. □

Whether the converse of the []-optional version of the /-right alternative of the ‖-left one of the last statement holds remains an open problem. On the other hand, the $\langle\rangle$-optional restriction by merely finitely-generated [P-]congruences of $\mathfrak{B}$ in (3.2) can not be omitted, even if P is equational, as it is demonstrated by the following apparently simplest counter-example:

Example 3.11. Let $\Sigma \triangleq (\Sigma_+ \cup 2 \cup \{\neg\})$, P the variety of De Morgan algebras and $(\mathfrak{A}|\mathfrak{C}) \in (\mathrm{Si} \,\|\, \mathrm{SI})(\mathsf{P})$ the Kleene|Boolean algebra with carrier $(A|C) \triangleq (2 \cup (\{\frac{1}{2}\}|\varnothing))$, in which case, by [22], P has an REDPC scheme, while $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, and so, by Lemma 2.9, there are a subdirect power $\mathfrak{B}$ of $\mathfrak{A}$ of degree $I \triangleq \omega$ and some $h \in \hom^{\mathrm{S}}(\mathfrak{B}, \mathfrak{C})$. Then, by (2.5) and the subdiract irreducibility of $\mathfrak{C}$, $(\ker h) \in \mathrm{MI}(\mathrm{Co}(\mathfrak{B}))$. Therefore, if the $\langle\rangle$-optional version of (3.2) held for arbitrary [P-]congruences of $\mathfrak{B}$, then, by the simplicity of $\mathfrak{A}$, there would be some $J \subseteq I$ such that $(\ker h) = (B^2 \cap (\bigcap_{i\in J}(\ker(\pi_i\restriction B))))$, in which case there would be some $j \in J$ such that $(\ker h) = (\ker(\pi_j\restriction B))$, and so, by the Homomorphism Theorem, $h^{-1} \circ (\pi_j\restriction B)$ would be an isomorphism from $\mathfrak{C}$ onto $\mathfrak{A}$, contrary to the inequality $2 \neq 3$. □

Corollary 3.12. *Let $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ be a variety with an REDPC scheme, $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ a pre-variety, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}[, \mathfrak{D}(, \mathfrak{E})] \in \mathsf{V}$, $\mathfrak{F} \in \mathsf{P}$ a finite subalgebra of $(\mathfrak{A}[(\times\mathfrak{E})]) \times \mathfrak{C}$ and [both] $(h[|g]) \in \hom_{[|(\mathrm{I})]}(\mathfrak{A}[|\mathfrak{C}], \mathfrak{B}[|\mathfrak{D}])$ Assume both $\mathrm{Co}(\mathfrak{A}\|\mathfrak{C}[(\|\mathfrak{E})]) = \{\Delta_{A\|C[(\|E)]}, (A\|C[(\|E)])^2\} \cup (\{\ker h\}\|(\varnothing[\cup\{\ker g\}])[(\|\varnothing)])$, while $(\{(\pi_0^{2(+1)}\restriction F)[\mathfrak{F}][, ((\pi_0^{2(+1)}\restriction F) \odot (($*

$\pi^{2(+1)}_{1(+1)}\restriction F)\circ g))[\mathfrak{F}](,((\pi^3_0\restriction F)\odot(\pi^3_1\restriction F))[\mathfrak{F}])]\}\cap\mathsf{P})=\varnothing$ {*cf.* (2.7)}, *whereas* $\mathfrak{F}\not\in((\mathfrak{B}[(\times\mathfrak{E})])\times\mathfrak{C})$. *Then,* $\mathfrak{F}\in\mathrm{SI}_{\mathsf{P}}(\mathsf{P})$.

Proof. Then, by (2.1), $\{\ker(\pi^{2(+1)}_0\restriction F)\cap\theta\mid\theta\in\{F^2[,\ker((\pi^{2(+1)}_{1(+1)}\restriction F)\circ g)(,\ker(\pi^3_1\restriction F))]\}\}$ is disjoint with $\Theta\triangleq\mathrm{Co}_{\mathsf{P}}(\mathfrak{F})\supseteq\Theta'\triangleq(\Theta\setminus\{\Delta_F\})\not\ni\Delta^F=(\ker(\pi^{2(+1)}_0\restriction F)[(\cap\ker(\pi^3_1\restriction F))]\cap\ker(\pi^{2(+1)}_{1(+1)}\restriction F))\in\Theta$, in which case, by the finiteness of $\mathfrak{F}\in\mathsf{V}$ and Theorem 3.10(3.2) [(as well as the injectivity of g)], $\Theta'\subseteq\wp(\ker((\pi^{2(+1)}_0\restriction F)\circ h)[(\cap\ker(\pi^3_1\restriction F))]\cap\ker(\pi^{2(+1)}_{1(+1)}\restriction F),F^2)$, and so, for any $\Theta''\subseteq\Theta'$, $(F^2\cap(\bigcap\Theta''))\supseteq(\ker((\pi^{2(+1)}_0\restriction F)\circ h)[(\cap\ker(\pi^3_1\restriction F))]\cap\ker(\pi^{2(+1)}_{1(+1)}\restriction F))\neq\Delta_F$, for, otherwise, by (2.1) and (2.7), $(((\pi^{2(+1)}_0\restriction F)\circ h)(\odot(\pi^{2(+1)}_1\restriction F))\odot(\pi^{2(+1)}_{1(+1)}\restriction F)$ would be an embedding of $\mathfrak{F}$ into $(\mathfrak{B}(\times\mathfrak{E}))\times\mathfrak{C}$. □

3.1.1.1. Implicativity versus filtrality.

Theorem 3.13. *Any [quasi-]equational/ [pre-]variety* $\mathsf{P}\subseteq\mathsf{A}_\Sigma$ *is implicative iff it is [relatively] /"both subdirectly-representable and" (subdirectly) /"finitely|principally" filtral.*

Proof. The "only if" part is by Theorem 3.10. Conversely, assume P is [relatively] /"both subdirectly-representable and" subdirectly principally filtral, in which case, by "Remark 2.4 as well as"/ Footnote 1, Lemma 2.5 and Corollary 2.10, P is [relatively] both subdirectly-representable and semi-simple with abstract and non-trivially-hereditary $\mathsf{K}\triangleq(\mathrm{Si}\,\|\,\mathrm{SI})_{[\mathsf{P}]}(\mathsf{P})$. Let $I\triangleq\{\theta\in\mathrm{Co}_{\mathsf{K}}(\mathfrak{Tm}^4_\Sigma)\mid(x_0\;\theta\;x_1)\Rightarrow(x_2\;\theta\;x_3)\}$, $\overline{\mathfrak{A}}\triangleq\langle\mathfrak{A}/i\rangle_{i\in I}\in\mathsf{K}^I$, $\mathfrak{D}\triangleq(\prod\overline{\mathfrak{A}})$, $h\triangleq(\prod_{i\in I}\nu_i)$ and $\bar a\triangleq\langle h(v_j)\rangle_{j\in4}$, in which case, by (2.2) and (2.7), $h\in\hom(\mathfrak{Tm}^4_\Sigma,\mathfrak{D})$, while $\mathfrak{B}\triangleq(\mathfrak{D}\restriction(\mathrm{img}\,h))$ is a subdirect product of $\overline{\mathfrak{A}}$, whereas $h\in\hom^{\mathrm{S}}(\mathfrak{Tm}^4_\Sigma,\mathfrak{B})$, and so $\vartheta\triangleq\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{P}]}(\langle a_0,a_1\rangle)=\theta^B_{\mathcal{F}}$, for some $\mathcal{F}\in\mathrm{Fi}(I)$. Then, $\langle a_0,a_1\rangle\in\vartheta$, in which case $\varepsilon_I(\langle a_2,a_3\rangle)\supseteq\varepsilon_I(\langle a_0,a_1\rangle)\in\mathcal{F}$, and so $\varepsilon_I(\langle a_2,a_3\rangle)\in\mathcal{F}$, i.e., $\langle a_2,a_3\rangle\in\vartheta$. Let $\mho\triangleq(\ker h)\subseteq\mathrm{Eq}^4_\Sigma$. Consider any $\mathfrak{C}\in\mathsf{K}$ and $g\in\hom(\mathfrak{Tm}^4_\Sigma,\mathfrak{C})$. Then, providing $\mho\subseteq\eta\triangleq(\ker g)\ni\langle x_0,x_1\rangle$, by the Homomorphism Theorem, $f\triangleq(h^{-1}\circ g)\in\hom(\mathfrak{B},\mathfrak{C})$, in which case, by (2.5), $\langle a_0,a_1\rangle\in\zeta\triangleq(\ker f)=f_*^{-1}[\Delta_C]\in\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{B})$, and so $\langle a_2,a_3\rangle\in\vartheta\subseteq\zeta$. In that case, $\langle x_2,x_3\rangle\in\eta$. Now, assume $(\langle x_0,x_1\rangle\in\eta)\Rightarrow(\langle x_2,x_3\rangle\in\eta)$, in which case $\mho\subseteq\eta$, i.e., $\mathfrak{C}\models(\bigwedge\mho)[g]$, whenever $\eta=\mathrm{Eq}^4_\Sigma$. Otherwise, by the {}-optional version of the right alternative of (2.4), $\eta\in I$, in which case, by (2.1), $\mho\subseteq\eta$, i.e., $\mathfrak{C}\models(\bigwedge\mho)[g]$, and so $\mho$ is an implicative system for K. Thus, P, being [relatively] subdirectly-representable, is $\mho$-implicative. □

This relativezes [7].

3.1.2. *Generic identity equivalence implication schemes for distributive lattice expansions.* Here, it is supposed that $\Sigma_+\subseteq\Sigma$. Given any $\mathfrak{A}\in\mathsf{A}_\Sigma$, $X\subseteq A$ and $\Omega\subseteq\mathrm{Tm}^1_\Sigma$, we have $\Omega^{\mathfrak{A}}_X:A\to\wp(\Omega),a\mapsto\{\varphi\in\Omega\mid\varphi^{\mathfrak{A}}(a)\in X\}$.

Given any $\bar\varphi\in(\mathrm{Tm}^1_\Sigma)^*$ with $x_0\in\Xi\triangleq(\mathrm{img}\,\bar\varphi)$, $\iota\in\Omega\in\wp(V_1,\Xi)$, $i\in2$ and $\Delta\in\wp(\Xi)$, let $\varepsilon^{i,\iota}_{\bar\varphi,\Delta}\triangleq((\wedge_+\langle(\bar\varphi\cap\Delta)*((\bar\varphi\cap\Delta)\circ[x_0/x_1]),\iota(x_{2+i})\rangle)\lessapprox(\vee_+\langle(\bar\varphi\setminus\Delta)*((\bar\varphi\setminus\Delta)\circ[x_0/x_1]),\iota(x_{3-i})\rangle))\in\mathrm{Eq}^4_\Sigma$ and $\mho^{\bar\varphi}_\Omega\triangleq\{\varepsilon^{i,\iota}_{\bar\varphi,\Delta}\mid i\in2,\iota\in\Omega,\Delta\in\wp(\Xi)\}\in\wp_\omega(\mathrm{Eq}^4_\Sigma)$.

Lemma 3.14. *Let* $\mathfrak{A}$ *be a* Σ*-algebra with (distributive) lattice* Σ_+*-reduct,* $\bar\varphi\in(\mathrm{Tm}^1_\Sigma)^*$ *with* $x_0\in\Xi\triangleq(\mathrm{img}\,\bar\varphi)$ *and* $\Omega\in\wp(V_1,\Xi)$. *Then,* $\mho^{\bar\varphi}_\Omega$ *is an identity reflexive symmetric (transitive implication) scheme for* $\mathfrak{A}$.

Proof. Clearly, for all $j\in2$, $\iota\in\Xi$ and $\Delta\in\wp(\Xi)$, there are some $\phi,\psi,\xi\in\mathrm{Tm}^3_\Sigma$ such that $(\varepsilon^{j,\iota}_{\bar\varphi,\Delta}[x_3/x_2])=((\phi\wedge\xi)\lessapprox(\psi\vee\xi))$, in which case this is satisfied in

lattice Σ-expansions, and so in $\mathfrak{A}$. Likewise, there are then some $\bar{\eta},\bar{\zeta}\in(\mathrm{Tm}^2_\Sigma)^+$ with $((\mathrm{img}\,\bar{\eta})\cap(\mathrm{img}\,\bar{\zeta}))\neq\varnothing$ such that $(\varepsilon^{j,\iota}_{\bar{\varphi},\Delta}[x_{2+i}/x_i]_{i\in 2})=((\wedge_+\bar{\eta})\lessapprox(\vee_+\bar{\zeta}))$, in which case this is satisfied in lattice Σ-expansions, and so in $\mathfrak{A}$. Furthermore, $(\mathcal{U}^{\bar{\varphi}}_\Omega[x_2/x_3,x_3/x_2])=\mathcal{U}^{\bar{\varphi}}_\Omega$. (Next, since the Σ_+-quasi-identity $\{(x_0\wedge x_1)\lessapprox(x_2\vee x_3),(x_0\wedge x_3)\lessapprox(x_2\vee x_4)\}\to((x_0\wedge x_1)\lessapprox(x_2\vee x_4))$, being satisfied in distributive latices, is so in $\mathfrak{A}$, so are logical consequences of its substitutional Σ-instances $(\mathcal{U}^{\bar{\varphi}}_\Omega\cup(\mathcal{U}^{\bar{\varphi}}_\Omega[x_{2+i}/x_{3+i}]_{i\in 2}))\to\Psi$, where $\Psi\in(\mathcal{U}^{\bar{\varphi}}_\Omega[x_3/x_4])$. Finally, consider any $a\in A$ and $\bar{b}\in(A^2\setminus\Delta_A)$, in which case, by the Prime Ideal Theorem, there are some $k\in 2$ and some prime filter F of $\mathfrak{A}$ such that $b_k\in F\not\ni b_{1-k}$, and so, as $\Delta\triangleq\Xi^{\mathfrak{A}}_F(a)\in\wp(\Xi)$ and $x_0\in\Omega$, $\mathfrak{A}\not\models(\bigwedge\mathcal{U}^{\bar{\varphi}}_\Omega)[x_i/a,x_{2+i}/b_i]_{i\in 2}$, for $\mathfrak{A}\not\models\varepsilon^{k,x_0}_{\bar{\varphi},\Delta}[x_i/a,x_{2+i}/b_i]_{i\in 2}$.) □

This, by Corollary 3.4, immediately yields:

Corollary 3.15. *Let $\mathfrak{A}$ be a non-one-element Σ-algebra with distributive lattice Σ_+-reduct, $\bar{\varphi}\in(\mathrm{Tm}^1_\Sigma)^*$ with $x_0\in\Xi\triangleq(\mathrm{img}\,\bar{\varphi})$ and $\Omega\in\wp(V_1,\Xi)$. Suppose $\mathcal{U}^{\bar{\varphi}}_\Omega$ is an implicative system for $\mathfrak{A}$. Then, $\mathfrak{A}$ is simple.*

3.1.2.1. Equality determinants versus implicativity. Recall that a *(logical) Σ-matrix* is any pair $\mathcal{A}=\langle\mathfrak{A},D\rangle$ with a Σ-algebra $\mathfrak{A}$ and a $D\subseteq A$, in which case an $\Omega\subseteq\mathrm{Tm}^1_\Sigma$ is called an *equality/identity determinant for $\mathcal{A}$*, if $\Omega^{\mathfrak{A}}_D$ is injective (cf. [20]), and so one *for* a class M of Σ-matrices, if it is so for each member of M.

Theorem 3.16. *Let M be a class of Σ-matrices and $\bar{\varphi}\in(\mathrm{Tm}^1_\Sigma)^*$ with $x_0\in\Xi\triangleq(\mathrm{img}\,\bar{\varphi})$. Suppose, for all $\mathcal{A}\in\mathsf{M}$, $\pi_0(\mathcal{A})\restriction\Sigma_+$ is a distributive lattice with set of its prime filters $\pi_1[\mathsf{M}\cap\pi_0^{-1}[\{\pi_0(\mathcal{A})\}]]$. Then, Ξ is an equality determinant for M iff $\mathcal{U}^{\bar{\varphi}}_{V_1}$ is an implicative system for $(\mathbf{IS}_{[>1]}\{\mathbf{P}^{\mathrm{U}}\})\pi_0[\mathsf{M}]$ ([in which case its members are simple]).*

Proof. Let $\mathcal{A}=\langle\mathfrak{A},D\rangle\in\mathsf{M}$, $\bar{a}\in A^2$ and, for any $\bar{b}\in A^2$, $h_{\bar{b}}\triangleq[x_i/a_i,x_{2+i}/b_i]_{i\in 2}$. First, assume Ξ is an equality determinant for M. Consider any $\bar{b}\in A^2$. Assume $\mathfrak{A}\not\models\varepsilon^{j,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for some $j\in 2$ and $\Delta\subseteq\Xi$, in which case, by the Prime Ideal Theorem, $\exists\mathcal{B}=\langle\mathfrak{A},D'\rangle\in\mathsf{M}:\forall k\in 2:\Delta=\Xi^{\mathfrak{A}}_{D'}(a_k)$, and so $a_0=a_1$. Then, by Lemma 3.14 with $\Omega=\Xi$, $\mathcal{U}^{\bar{\varphi}}_{V_1}$ is an implicative system for $\mathfrak{A}$. Conversely, assume $\mathcal{U}^{\bar{\varphi}}_{V_1 r}$ is an implicative system for $\mathfrak{A}$ and $\Delta\triangleq\Xi^{\mathfrak{A}}_D(a_0)=\Xi^{\mathfrak{A}}_D(a_1)$. Take any $\bar{b}\in(D\times(A\setminus D))\neq\varnothing$, in which case, as $\Delta\subseteq\Xi\ni x_0$, $\mathfrak{A}\not\models\varepsilon^{0,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for D is a prime filter of $\mathfrak{A}\restriction\Sigma_+$, and so $a_0=a_1$. (Finally, Corollary 3.15 completes the argument.) □

3.2. **Disjunctivity.** Unless otherwise specified, fix any $\mathcal{U}\subseteq\mathrm{Eq}^4_\Sigma$.

3.2.1. *Disjunctivity versus finite subdirect irreducibility and congruence-distributivity.*

Lemma 3.17. *Any $\mathcal{U}$-disjunctive /finite non-one-element $\mathfrak{A}\in\mathsf{A}_\Sigma$ is finitely/ subdirectly-irreducible. In particular, any disjunctive pre-variety is (relatively) finitely subdirectly-representable.*

Proof. Consider any $\theta,\vartheta\in(\mathrm{Co}(\mathfrak{A})\setminus\{\Delta_A\})$ and take any $(\bar{a}|\bar{b})\in((\theta|\vartheta)\setminus\{\Delta_A\})\neq\varnothing$, in which case the Σ-identities in $\mathcal{U}[x_{1|3}/x_{0|2}]$, being true in $\mathfrak{A}$, are so in $\mathfrak{A}/(\theta|\vartheta)$ (in particular, under $[x_{0|2}/\nu_{\theta|\vartheta}((a|b)_0),x_{(2|0)+i}/\nu_{\theta|\vartheta}((b|a)_i)]_{i\in 2}$), and so $\Delta_A\not\supseteq\{\langle\phi^{\mathfrak{A}}[x_i/a_i,x_{2+i}/b_i]_{i\in 2},\phi^{\mathfrak{A}}[x_i/a_i,x_{2+i}/b_i]_{i\in 2}\rangle\mid(\phi\approx\psi)\in\mathcal{U}\}\subseteq(\theta\cap\vartheta)$. Then, $(\theta\cap\vartheta)\neq\Delta_A$. Thus, induction on the cardinality of finite subsets of $\mathrm{Co}(\mathfrak{A})$ ends the proof. □

Lemma 3.18. *Let $\mathsf{P}\subseteq\mathsf{A}_\Sigma\ni\mathfrak{A}$ be a $\mathcal{U}$-disjunctive pre-variety and $X,Y,Z\subseteq A^2$. Then, $\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\mathcal{U}^{\mathfrak{A}}(X,Y)\cup Z)=(\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(X\cup Z)\cap\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(Y\cup Z))$.*

Proof. In that case, P is generated by $\mathsf{K} \triangleq \mathsf{P}_\mathcal{U} = \mathbf{IS}\mathsf{K}$, so, by Remark 2.2 and (2.8), $\mathrm{Co}_\mathsf{K}(\mathfrak{A})$ is a basis of $\mathrm{Co}_\mathsf{P}(\mathfrak{A})$. Then, for any $\theta \in \mathrm{Co}_\mathsf{K}(\mathfrak{A})$, $\mathfrak{A}/\theta$ is $\mathcal{U}$-disjunctive, in which case $(\mathcal{U}^\mathfrak{A}(X,Y) \cup Z) \subseteq \theta$ iff either $(X \cup Z) \subseteq \theta$ or $(Y \cup Z) \subseteq \theta$, and so, for any $\bar{a} \in A^2$, $(\bar{a} \in \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\mathcal{U}^\mathfrak{A}(X,Y) \cup Z)) \Leftrightarrow (\forall \theta \in \mathrm{Co}_\mathsf{K}(\mathfrak{A}) : ((\mathcal{U}^\mathfrak{A}(X,Y) \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow ((\forall \theta \in \mathrm{Co}_\mathsf{K}(\mathfrak{A}) : (X \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta))\&(\forall \theta \in \mathrm{Co}_\mathsf{K}(\mathfrak{A}) : ((Y \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow (\bar{a} \in (\mathrm{Cg}^\mathfrak{A}_\mathsf{P}(X \cup Z) \cap \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(Y \cup Z)))$, as required. □

Corollary 3.19. *Any $\mathcal{U}$-disjunctive [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is [relatively] congruence-distributive, and so is any [quasi-equational/finitely] implicative one.*

Proof. Then, by Lemma 3.18, for any $\mathfrak{A} \in \mathsf{P}$ and $\theta, \vartheta, \eta \in \mathrm{Co}_\mathsf{P}(\mathfrak{A})$, we have $(\mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\theta \cup \eta) \cap \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\vartheta \cup \eta)) = \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\mathcal{U}^\mathfrak{A}(\theta,\vartheta) \cup \eta) = \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\mathcal{U}^\mathfrak{A}(\theta,\vartheta)) \cup \eta) = \mathrm{Cg}^\mathfrak{A}_\mathsf{P}((\mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\theta) \cap \mathrm{Cg}^\mathfrak{A}_\mathsf{P}(\vartheta)) \cup \eta) = \mathrm{Cg}^\mathfrak{A}_\mathsf{P}((\theta \cap \vartheta) \cup \eta)$, as required. □

Lemma 3.20. *Let $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ be a $\mathcal{U}$-implicative pre-variety and $\mathcal{U}'$ a disjunctive system for $\mathsf{P}_\mathcal{U}$. Then, every $\mathcal{U}'$-disjunctive member of P is $\mathcal{U}$-implicative.*

Proof. In that case, $\mathcal{U}$, being is an identity implication scheme for $\mathsf{P}_\mathcal{U}$, is so for $\mathsf{P} = \mathbf{ISP}\mathsf{P}_\mathcal{U}$, while the Σ-identities in $\bigcup\{\mathcal{U}'[x_{2+i}/\varphi_i]_{i\in 2} \mid \bar{\varphi} \in \mathcal{U}\}$, being true in $\mathsf{P}_\mathcal{U}$, are so in P, and so $\mathcal{U}'$-disjunctive members of P are $\mathcal{U}$-implicative, as required. □

Corollary 3.21. *For any $\mathcal{U}$-disjunctive [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, $\mathsf{P}_\mathcal{U} = (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{\cong 1}_\Sigma)$. In particular, any [quasi-equational/finitely] implicative [pre-]variety is [relatively] finitely semi-simple.*

Proof. Then, any one-element Σ-algebra is $\mathcal{U}$-disjunctive, while, for any $\mathfrak{A} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$ and $\bar{a}, \bar{b} \in (A^2 \setminus \Delta_A)$, since $\mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\bar{a}|\bar{b}) \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$, whereas, by Lemma 3.18, $(\mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\bar{a}) \cap \mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\bar{b})) = \mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\mathcal{U}^\mathfrak{A}(\bar{a}|\bar{b}))$, we have $\mathcal{U}^\mathfrak{A}(\bar{a}|\bar{b}) \neq \Delta_A = \mathrm{Cg}^\mathfrak{A}_{[\mathsf{P}]}(\Delta_A)$, i.e., $\mathfrak{A} \not\models (\bigwedge \mathcal{U})[x_i/a_i, x_{2+i}/b_i]_{i\in 2}$, in which case $\mathfrak{A}$ is $\mathcal{U}$-disjunctive, because the Σ-identities in $\bigcup_{j\in 2} \mathcal{U}[x_{(2\cdot j)}/x_{(2\cdot j)+1}]$, being true in $\mathsf{P}_\mathcal{U}$, are so in $\mathbf{ISP}\mathsf{P}_\mathcal{U} = \mathsf{P} \ni \mathfrak{A}$, and so Lemmas 3.4, 3.17, 3.20 and [21, Remark 2.4] complete the argument. □

Theorem 3.22. *Any [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ is disjunctive iff it is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{\cong 1}_\Sigma$ being "a universal (infinitary) model class"/hereditary.*

Proof. The "only if" part is by Lemma 3.2.1 and Corollary 3.21. Conversely, assume P is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with hereditary $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{\cong 1}_\Sigma$, in which case, by Remark 2.4, it is [relatively] finitely-subdirectly-representable, while, by (2.5), $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)$, where $\theta \triangleq (\mathrm{Eq}^4_\Sigma \cap (\bigcap \mathrm{Co}_{\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})}(\mathfrak{Tm}^4_\Sigma))) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma)$, is distributive, for $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma/\theta)$ is so. Let $\forall j \in 2 : \vartheta_j \triangleq \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\theta \cup \{\langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle\}) \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)) \ni \mathcal{U} \triangleq (\vartheta_0 \cap \vartheta_1) \subseteq \mathrm{Eq}^4_\Sigma$. Consider any $\mathfrak{A} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$ and any $\bar{a} \in A^4$. Let $h \in \hom(\mathfrak{Tm}^4_\Sigma, \mathfrak{A})$ extend $\{\langle x_i, a_i\rangle \mid i \in 4\}$, in which case $\mathfrak{B} \triangleq (\mathfrak{A}\restriction(\mathrm{img}\, h)) \in (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathsf{A}^{\cong 1}_\Sigma)$, and so $((\{\langle a_0, a_1\rangle, \langle a_2, a_3\rangle\} \cap \Delta_A) \neq \varnothing)$ &$\restriction\Leftrightarrow (\mathfrak{A} \models \Phi^4_\mathcal{U}[h\restriction V_4])$, unless $\mathfrak{B} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$. Otherwise, by (2.5) and the Homomorphism Theorem, $\theta \subseteq \eta \triangleq (\ker h) \in \mathrm{MI}^\omega(\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma))$, in which case we have:

$$(\mathfrak{A} \models \Phi^4_\mathcal{U}[h\restriction V_4]) \Leftrightarrow ((\vartheta_0 \cap \vartheta_1) = \mathcal{U} \subseteq \eta) \Leftrightarrow (\eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup (\vartheta_0 \cap \vartheta_1)) =$$
$$(\mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_0) \cap \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_1)) \Leftrightarrow (\exists j \in 2 : \eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_j)) \Leftrightarrow$$
$$(\exists j \in 2 : \vartheta_j \subseteq \eta) \Leftrightarrow (\exists j \in 2 : \langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle \in \eta) \Leftrightarrow (\exists j \in 2 : a_{2\cdot j} = a_{(2\cdot j)+1}),$$

and so $\mathcal{U}$ is a disjunctive system for $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$. Thus, P, being [relatively] finitely-subdirectly-representable, is $\mathcal{U}$-disjunctive, as required. □

This, by Remark 2.4 and Corollary 3.21 (as well as the Compactness Theorem for ultra-multiplicative classes; cf., e.g., [15]), immediately yields:

Corollary 3.23. *Any [quasi-]variety* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *is (finitely) disjunctive iff it is [relatively] congruence-distributive with* $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}^{\cong 1}_{\Sigma}$ *being "a universal (first-order) model class"/"hereditary (and ultra-multiplicative)".*

This, in its turn, by Footnote 1, Corollary 2.11 and Lemma 2.12, immediately yields:

Corollary 3.24. *Any locally-finite [quasi-]variety* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *is (finitely) disjunctive iff it is [relatively] congruence-distributive with* $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}^{\cong 1}_{\Sigma}$ *being "a universal* {*infinitary*} *model class"/hereditary.*

Finally, this, by the congruence-distributivity of lattice expansions (cf., e.g., [17]) and Corollary 2.7, immediately yields:

Corollary 3.25. *Suppose* $\Sigma_+ \subseteq \Sigma$*. Then, any finitely-generated variety* $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ *of lattice expansions with non-trivially-hereditary* $\mathrm{SI}^{(\omega)}(\mathsf{V})$ *is finitely disjunctive.*

This provides an immediate (though far from being constructive) insight into the finite disjunctivity of the finitely-generated variety of distributive/Stone|"De Morgan" lattices/algebras|algebras‖lattices, a constructive one being given by [19, Example 1/2] and [20, Lemma 11].

3.2.1.1. Implicativity versus finite semi-simplicity and disjunctivity. By Footnote 1, Theorem 3.13, Corollaries 2.8, 2.11, 2.13, 3.21, 3.23, 3.24, Lemma 2.12 and [21, Remark 2.4], we eventually get:

Theorem 3.26. *Any locally-finite/ [quasi-]variety* $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *is implicative iff it is /finitely both disjunctive and [relatively] semi-simple iff it is [relatively] both congruence-distributive and semi-simple with* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathsf{A}^{\cong 1}_{\Sigma}$ *being "a universal /first-order model class"| "hereditary /"and ultra-multiplicative"".*

This, by the congruence-distributivity of lattice expansions (cf., e.g., [17]), Corollaries 2.7, 3.4 and Footnote 1, immediately yields:

Corollary 3.27. *Suppose* $\Sigma_+ \subseteq \Sigma$*. Then, any locally-finite variety* $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ *of lattice expansions is implicative iff it is semi-simple "and (finitely) disjunctive"| "with non-trivially-hereditary* $(\mathrm{Si} \,|\, \mathrm{SI})(\mathsf{V})$*".*

Corollary 3.28. *Suppose* $\Sigma_+ \subseteq \Sigma$*. Let* $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ *be a finite set of finite lattice expansions without non-simple non-one-element subalgebras and* V *the variety generated by* K*. Then,* V *is implicative with* $(\mathrm{Si} \,|\, \mathrm{SI})(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{K}$.

These provide an immediate /{though far from being constructive} insight into the not/ implicativity of (and so not/ REDPC for; cf. Theorem 3.7) the not/ semi-simple finitely-generated variety of Stone/distributive|"De Morgan" algebras/lattices|algebras‖lattices /(cf. [9]|[22]‖) /"a constructive one being given by Theorem 3.16 and [19, Example 1]|"Remark 4.3"".

Whether the /-alternative stipulations are necessary in Theorem 3.26 remains an open issue. On the other hand, the necessity of the "[relative] congruence-distributivity"//"lattice expansion" stipulation therein// as well as in Corollaries 3.23, 3.24, 3.25, 3.27, 3.28 and Theorem 3.22 is demonstrated by:

Example 3.29. Let $\Sigma = \{\wedge\}$ and SL the variety of semi-lattices, in which case, for any filter $F \neq A$ of any $\mathfrak{A} \in \mathsf{SL}$, χ^F_A is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{S}_2 \in \mathsf{SL}$ with $S_2 \triangleq 2$ and $\wedge^{\mathfrak{A}} \triangleq (\cap\restriction 2^2)$, and so, by (2.8), $\mathsf{SL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{S}_2$. Now, assume $|A| > 2$, in which case, providing $\mathfrak{A}$ is a chain, for any $\bar{a} \in A^3$ with $|\operatorname{img} \bar{a}| = 3$ such that $a_0 \leqslant^{\mathfrak{A}} a_1 \leqslant^{\mathfrak{A}} a_2$ and $i \in 2$, $\Delta_A \neq \theta_i \triangleq ([a_i, a_{i+1}]^2_{\mathfrak{A}} \cup$

$\Delta_A) = \mathrm{Cg}^{\mathfrak{A}}(\{\langle a_i, a_{i+1}\rangle\}) \in \mathrm{Co}(\mathfrak{A})$, while $(\theta_0 \cap \theta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Otherwise, take any $\bar{b} \in A^2$ such that $c \triangleq (b_0 \wedge^{\mathfrak{A}} b_1) \notin (\mathrm{img}\,\bar{b})$, in which case, for each $j \in 2$, $\vartheta_j \triangleq ((\bigcup\{[c \wedge^{\mathfrak{A}} d, b_j \wedge^{\mathfrak{A}} d]^2_{\mathfrak{A}} \mid d \in A\}) \cup \Delta_A) \supsetneq \Delta_A$ is symmetric and forms a subalgebra of $\mathfrak{A}^2$, and so the transitive closure $\eta_j = \mathrm{Cg}^{\mathfrak{A}}(\{\langle c, b_j\rangle\}) \supseteq \vartheta_j$ of ϑ_j is a congruence of $\mathfrak{A}$ distinct from Δ_A. By contradiction, prove that $(\eta_0 \cap \eta_1) \subseteq \Delta_A$. For suppose $(\eta_0 \cap \eta_1) \nsubseteq \Delta_A$. Take any $\bar{e} \in ((\eta_0 \cap \eta_1) \setminus \Delta_A) \neq \varnothing$, in which case, for all $k, l \in 2$, $\langle e_k, e_{1-k}\rangle \in (\theta_l \setminus \Delta_A)$, that is, there are some $m_l \in \omega$, $\bar{f}^l \in A^{m_l+2}$ and $\bar{g}^l \in A^{m_l+1}$ such that $f^l_0 = e_k$, $f^l_{m_l+1} = e_{1-k}$ and, for every $n \in (m_l + 1)$, $f^l_{n[+1]} \in [c \wedge^{\mathfrak{A}} g^l_n, b_l \wedge^{\mathfrak{A}} g^l_n]_{\mathfrak{A}}$, and so $e_k \leqslant^{\mathfrak{A}} c$, when taking $n = 0$, because $\{l, 1-l\} = 2$, while $e_k = f^{l|(1-l)}_0 \leqslant^{\mathfrak{A}} (b_{l|(1-l)} \wedge^{\mathfrak{A}} g^{l|(1-l)}_0) \leqslant^{\mathfrak{A}} b_{l|(1-l)}$. By induction on any $\ell \in (m_l + 2)$, show that $e_k \leqslant^{\mathfrak{A}} f^l_\ell$. The case $\ell = 0$ is by the equality $e_k = f^l_0$. Otherwise, $(m_l + 2) \ni (\ell - 1) < \ell$, in which case, by induction hypothesis, we have $c \geqslant^{\mathfrak{A}} e_k \leqslant^{\mathfrak{A}} f^l_{\ell-1} \leqslant^{\mathfrak{A}} (b_l \wedge^{\mathfrak{A}} g^l_{\ell-1}) \leqslant^{\mathfrak{A}} g^l_{\ell-1}$, and so we get $e_k \leqslant^{\mathfrak{A}} (c \wedge^{\mathfrak{A}} g^l_{\ell-1}) \leqslant^{\mathfrak{A}} f^l_\ell$. In particular, $e_k \leqslant^{\mathfrak{A}} e_{1-k}$, when taking $\ell = (m_l + 1)$, since $f^l_{m_l+1} = e_{1-k}$. Then, $e_0 = e_1$, in which case this contradiction shows that $(\eta_0 \cap \eta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Thus, by (2.10) as well as the simplicity of two-element algebras and absence of their proper non-one-element subalgebras, $((\mathrm{SI}^{(\omega)} \,|\, \mathrm{Si})(\mathsf{SL})\{\cup \mathsf{A}^{=1}_\Sigma\}) = (\mathbf{I}\mathfrak{S}_2\{\cup \mathsf{A}^{=1}_\Sigma\})$ is the class of {no-more-than-}two-element semi-lattices {that is, the universal first-order model subclass of SL relatively axiomatized by the single universal first-order sentence $\forall_{\imath\in 3} x_\imath((x_2 \approx x_1) \vee (x_2 \approx x_0) \vee (x_1 \approx x_0))$}, while SL, being finitely-semi-simple and finitely-generated, is semi-simple and locally-finite. On the other hand, since $\mathrm{Fi}(2) = \{\wp(N, 2) \mid N \subseteq 2\}$, the set $\{\Delta_{2^2}, (2^2)^2\} \cup \{\ker(\pi_\jmath{\restriction}2^2) \mid \jmath \in 2\}$ of filtral congruences of $\mathfrak{S}^2_2$ does not contain its congruence $\Delta_{2^2} \cup \{\langle\langle 0, \Bbbk\rangle, \langle 0, 1 - \Bbbk\rangle\rangle \mid \Bbbk \in 2\}$, in which case, by Theorem 3.13, SL, not being filtral, is not implicative, and so, by Theorem 3.26, is neither congruence-distributive nor disjunctive. □

3.2.2. *Disjunctivity versus distributivity of lattices of sub-varieties.*

Lemma 3.30. *Let K be a class of Σ-algebras with a disjunctive system $\mathcal{U} \subseteq \mathrm{Eq}^4_\Sigma$ as well as R and S are relative sub-varieties of K. Then, so is $\mathsf{R} \cap \| \cup \mathsf{S}$. In particular, relative sub-varieties of K form a distributive lattice.*

Proof. Take any $\mathcal{I}, \mathcal{J} \subseteq \mathrm{Tm}^\omega_\Sigma$ with $(\mathsf{R}|\mathsf{S}) = (\mathsf{K} \cap \mathrm{Mod}(\mathcal{I}|\mathcal{J}))$, in which case $(\mathsf{R} \cap \| \cup \mathsf{S}) = (\mathsf{K} \cap \mathrm{Mod}((\mathcal{I} \cup \mathcal{J}) \| \bigcup\{\mathcal{U}[x_i/\phi_i, x_{2+i}/\psi_i]_{i\in 2} \mid (\bar{\phi}|\bar{\psi}) \in ((\mathcal{I}|\mathcal{J})[x_j/x_{(2\cdot j)+(0|1)}]_{j\in\omega})\}))$, and so the distributivity of unions with intersections completes the argument. □

This, by (2.10), (2.9) and Lemma 3.17, immediately yields:

Corollary 3.31. *Let K be a [finite] class of finite Σ-algebras with a disjunctive system $\mathcal{U} \subseteq \mathrm{Eq}^4_\Sigma$ and P the pre-variety generated by K. Suppose P is a variety. Then, $\mathrm{SI}(\mathsf{P}) = \mathbf{IS}_{>1}\mathsf{K}$, in which case $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the lattices of sub-varieties of P and relative ones of $\mathbf{S}_{\{>1\}}\mathsf{K}$, and so they are distributive [and finite].*

Likewise, by (2.10), (2.9), Theorem 3.7 (as well as [21, Remark 2.4] and Lemma 3.30), we immediately have:

Corollary 3.32. *Let K be a [finite] class of [finite] Σ-algebras with a (finite) implicative system $\mathcal{U} \subseteq \mathrm{Eq}^4_\Sigma$ and P the pre-variety generated by K. Suppose P is a variety. Then, $(\mathrm{SI}\,|\,\mathrm{Si})(\mathsf{P}) = \mathsf{P}^{\geqslant 1}_{\mathcal{U}} = \mathbf{IS}_{>1}\mathsf{K}$, in which case $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the [finite] (distributive) lattices of sub-varieties of P and relative ones of $\mathbf{S}_{\{>1\}}\mathsf{K}$.*

4. Morgan-Stone lattices versus distributive ones

From now on, we deal with the signatures $\Sigma^{(-)}_{+[,01]} \triangleq (\Sigma_+(\cup\{\neg\})[\cup\{\bot,\top\}])$, [bounded] {distributive} lattices being supposed to be $\Sigma_{+[,01]}$-algebras with their variety denoted by [B]{D}L and the chain [bounded] distributive lattice with carrier $n \in (\omega \setminus 2)$ and the natural ordering on this denoted by $\mathfrak{D}_{n[,01]}$, in which case $\epsilon^n_2 \triangleq \{\langle 0,0\rangle, \langle 1,n-1\rangle\}$ is an embedding of $\mathfrak{D}_{2[,01]}$ into $\mathfrak{D}_{n[,01]}$, while, for each $i \in 2$, $\epsilon^4_{3:i} \triangleq (\chi_3^{3\setminus(2-i)} \times \chi_3^{3\setminus(1+i)})$ is an embedding of $\mathfrak{D}_{3[,01]}$ into $\mathfrak{D}^2_{2[,01]}$. First, taking the Prime Ideal Theorem, (2.8), (2.10) and Corollary 3.14 into account, we immediately have the following well-known fact (cf. [9] as to REDPC for [B]DL):

Lemma 4.1. *Let $\mathfrak{A} \in$ [B]L and $F \subseteq A$. Suppose F is either a prime filter of $\mathfrak{A}$ or in $\{\varnothing, A\}$. Then, [unless $F \in \{\varnothing, A\}$] $h \triangleq \chi^F_A \in \hom(\mathfrak{A}, \mathfrak{D}_{2[,01]})$ [and $h[A] = 2$], in which case* [B]DL $= \mathbf{IP}^{\mathrm{SD}}\mathfrak{D}_{2[,01]}$*, and so* [B]DL *is the semi-simple [pre-/quasi-]variety generated by $\mathfrak{D}_{2[,01]}$ with* (Si | SI)([B]DL) $= \mathbf{I}\mathfrak{D}_{2[,01]}$ *and REDPC scheme $\mho^{\langle x_0\rangle}_{V_1}$.*

A *[bounded] (De) Morgan-Stone {(D)MS} lattice* is any $\Sigma^-_{+[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and which satisfies the Σ^-_+-identities:

$$\neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_1), \tag{4.1}$$

$$x_0 \lessapprox \neg\neg x_0, \tag{4.2}$$

in which case, by (4.1) [and (4.2)$[x_0/\top]$], it satisfies the Σ^-_+-quasi-identity [and the $\Sigma^-_{+[,01]}$-identity]:

$$(x_0 \lessapprox x_1) \to (\neg x_1 \lessapprox \neg x_0)[, \tag{4.3}$$

$$\neg\neg\top \approx \top], \tag{4.4}$$

and so the $\Sigma^-_{+[,01]}$-identities:

$$\neg(x_0 \vee x_1) \approx (\neg x_0 \wedge \neg x_1), \tag{4.5}$$

$$\neg\neg\neg x_0 \approx \neg x_0[, \tag{4.6}$$

$$\neg\bot \approx \top], \tag{4.7}$$

their variety being denoted by [B](D)MSL. Then, bounded Morgan-Stone lattices, satisfying the $\Sigma^-_{+,01}$-identity:

$$\neg\top \approx \bot, \tag{4.8}$$

are nothing but *(De) Morgan-Stone {MS} algebras* [3] ⟨cf. [24]⟩, their variety being denoted by (D)MSA. An $a \in A$ is called {a} *(negatively-)idempotent {element of* an $\mathfrak{A} \in$ MSL*}*, if $\{(\neg^{\mathfrak{A}})a\}$ forms a subalgebra of $\mathfrak{A}[\restriction\Sigma^-_+]$, i.e., $\neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})a = (\neg^{\mathfrak{A}})a$, with their set denoted by $\Im^{\mathfrak{A}}_{(\neg)}$, [bounded] Morgan-Stone lattices with carrier of cardinality no less than $2(\{-1\})$ and with({out non-}negatively-)idempotent elements being said to be *({totally} negatively-)idempotent*.

Remark 4.2. By (4.1), (4.5), (4.6), Corollary 3.14 and Theorem 3.5, $\mho^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}_{\{x_0,\neg x_0,\{\neg\neg x_0\}}$ is an REDPC scheme for [B]MS(L[/A]). □

4.1. **Subdirectly-irreducibles.** Let $\mathfrak{MS}_6$ be the Σ^-_+-algebra with $(\mathfrak{MS}_6\restriction\Sigma^-_+) \triangleq ((\mathfrak{D}^2_2\restriction(2^2 \setminus \{\langle 1,0\rangle\})) \times \mathfrak{D}_2)$ and $\neg^{\mathfrak{MS}_6}\bar{a} \triangleq \langle 1-a_2, 1-a_2, 1-a_1\rangle$, for all $\bar{a} \in MS_6$ (the Hasse diagram of its lattice reduct with its [non-]idempotent elements marked by [non-]solid circles and arrows reflecting action of its operation $\neg$ on its non-idempotent elements is depicted at Figure 1), in which case it is routine to check to be a Morgan-Stone lattice, and so are both $\mathfrak{MS}_{5(:0)} \triangleq (\mathfrak{MS}_6\restriction(MS_6 \setminus \{\langle 0,0,1\rangle\})$

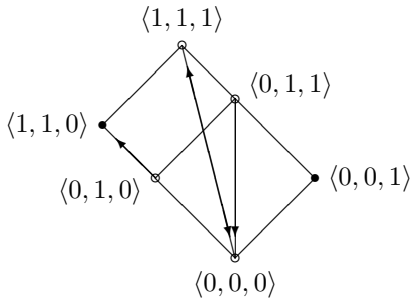

FIGURE 1. The Morgan-Stone lattice $\mathfrak{MS}_6$.

and $\mathfrak{MS}_{2(:0)} \triangleq (\mathfrak{MS}_5\restriction\{\langle i,1,0\rangle \mid i \in 2\})$ as well as, for each $j \in 2$, $\mathfrak{MS}_{4:j} \triangleq (\mathfrak{MS}_{5+j}\restriction(MS_{5+j} \setminus (((j+1)\times\{1\})\times\{1-j\})))$. Likewise, let $(((\mathfrak{D})\mathfrak{M})|\mathfrak{S})_{4|3}$ be the Σ^-_+-algebra with $(((\mathfrak{M}|\mathfrak{S})_{4|3}\restriction\Sigma^-_+) \triangleq \mathfrak{D}^{2|}_{2|3}$ and $\neg^{(\mathfrak{M}|\mathfrak{S})_{4|3}} \triangleq ((((\pi_1\restriction 2)\circ(2^2\setminus\Delta_2))\times((\pi_0\restriction 2)\circ(2^2\setminus\Delta_2)))|\chi^1_3)$, in which case $\epsilon^{6|5}_{4|3} \triangleq ((((\pi_0\restriction 2^2)\times(\pi_0\restriction 2^2))\times(\pi_1\restriction 2^2))|(\epsilon^4_{3:0}\times\chi_3^{3\setminus 1}))$ is an embedding of $(\mathfrak{DM}|\mathfrak{S})_{4|3}$ into $(\mathfrak{MS}|\mathfrak{MS})_{6|5}$. Finally, for any $n \in (\{3,4\}|\{2\})$, let $(\mathfrak{K}|\mathfrak{B})_n$ be the Σ^-_+-algebra with $((\mathfrak{K}|\mathfrak{B})_n\restriction\Sigma^-_+) \triangleq \mathfrak{D}_n$ and $\neg^{(\mathfrak{K}|\mathfrak{B})_n} \triangleq \{\langle m, n-1-m\rangle \mid m \in n\}$, in which case $\epsilon_2^{3\|4}$ is an embedding of $\mathfrak{B}_2$ into $\mathfrak{K}_{3\|4}$, while, for every $l \in 2$, $\epsilon^4_{3:l}$ is an embedding of $\mathfrak{K}_3$ into $\mathfrak{DM}_4$, and so $\epsilon^4_{3:l}\circ\epsilon^6_4$ is that into $\mathfrak{MS}_{4:(1-l)}$. Moreover, $\{MS_6, MS_5, MS_2, \mathrm{img}(\epsilon^3_2\circ\epsilon^5_3)\}\cup(\bigcup\{\{MS_{4:k}, \mathrm{img}(\epsilon^4_{3:k}\circ\epsilon^6_4)\} \mid k \in 2\})$ are exactly the carriers of members of $\mathbf{S}_{>1}\mathfrak{MS}_6$, in which case these are isomorphic to those of the skeleton $\mathsf{MS} \triangleq (\{\mathfrak{MS}_\ell \mid \ell \in \{6,5,2\}\}\cup\{\mathfrak{MS}_{4:\Bbbk} \mid \Bbbk \in 2\}\cup\{\mathfrak{DM}_4, \mathfrak{K}_3, \mathfrak{S}_3, \mathfrak{B}_2\})$, and so this is that of $\mathbf{IS}_{>1}\mathfrak{MS}_6$ with the embeddability *partial* ordering $\preceq$ between members of MS, for these are all finite. And what is more, $D_6 \triangleq (MS_6 \cap \pi_0^{-1}[\{1\}])$ is a prime filter of $\mathfrak{MS}_6\restriction\Sigma_+$, while $\Omega \triangleq \{x_0, \neg x_0, \neg\neg x_0\}$ is an equality determinant for $\langle\mathfrak{MS}_6, D_6\rangle$, in which case, by [20, Lemma 11], $\mho_\Omega \triangleq \{(\tau(x_\imath)\wedge\rho(x_{2+\jmath})) \lessapprox (\tau(x_{1-\imath})\vee\rho(x_{3-\jmath})) \mid \imath,\jmath \in 2, \tau,\rho \in \Omega\}$ is a disjunctive system for $\mathfrak{MS}_6$, and so, for $\mathbf{IS}\mathfrak{MS}_6$.

Remark 4.3. Elements of $\mathcal{PF}_4 \triangleq \{2^2 \cap \pi_i^{-1}[\{1\}] \mid i \in 2\}$ are exactly all prime filters of $\mathfrak{D}^2_2$, while $\{x_0, \neg x_0\}$ is an equality determinant for $\mathsf{M} \triangleq (\{\mathfrak{DM}_4\}\times\mathcal{PF}_4)$, in which case, by Theorem 3.16, $\mho^{\langle x_0,\neg x_0\rangle}_{V_1}$ is an implicative system for $\mathbf{IS}_{\{>1\}}\mathfrak{DM}_4$ {and so, by Corollary 3.15, its members are simple, as it is well-known but shown directly in a more cumbersome way}. □

Theorem 4.4. *For any prime filter F of the Σ_+-reduct of any $\mathfrak{A} \in \mathsf{MSL}$ there is an $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ with $(\ker h) \subseteq (\ker \chi^F_A)$, in which case MSL is the [pre-/quasi-]variety generated by $\mathfrak{MS}_6$ with REDPC scheme $\mho^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}_\Omega$, and so* $\mathrm{SI}(\mathsf{MSL}) = \mathbf{I}\mathsf{MS}$.

Proof. Let $f \triangleq \chi^F_A$, $G \triangleq (\neg^{\mathfrak{A}})^{-1}[(\neg^{\mathfrak{A}})^{-1}[F]]$, $H \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[F])$ and $h \triangleq (f \odot \chi^G_A) \odot \chi^H_A)$, in which case, by (2.1) and (4.6), $(\ker f) \supseteq (((\ker f)\cap(\ker\chi^G_A))\cap(\ker\chi^H_A)) = (\ker h) \subseteq (\neg^{\mathfrak{A}}\circ h)$, while, by (4.1) and (4.5), $G|H$ is either a prime filter of $\mathfrak{A}\restriction\Sigma_+$ or in $\{\varnothing, A\}$, whereas, by (4.2), $F \subseteq G$, and so, by (2.2), $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$. Then, by (2.7), Lemma 4.1 and the Homomorphism Theorem, h is a surjective homomorphism from $\mathfrak{A}$ onto the Σ^-_+-algebra $\mathfrak{B}$ with $(\mathfrak{B}\restriction\Sigma_+) \triangleq (\mathfrak{D}^3_2\restriction h[A])$ as well as $\neg^{\mathfrak{B}} \triangleq (h^{-1}\circ\neg^{\mathfrak{A}}\circ h)$, in which case $B \subseteq MS_6$, since $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$, and so $\mathfrak{B} = (\mathfrak{MS}_6\restriction h[A])$, as, for all $a \in A$, $(\neg^{\mathfrak{A}}a \in G) \Leftrightarrow (\neg^{\mathfrak{A}}a \in F) \Leftrightarrow (a \notin H)$, in view of (4.6), as well as $(\neg^{\mathfrak{A}}a \in H) \Leftrightarrow (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \notin F) \Leftrightarrow (a \notin G)$. Hence, $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ and $(\ker h) \subseteq (\ker f)$.

Thus, the Prime Ideal Theorem, (2.8), Corollary 3.31 and Remark 4.2 complete the argument. □

The Σ^-_+-reduct of any $\mathfrak{A} \in \mathsf{MS}$, being a finite lattice, has zero/unit a/b, in which case we have the bounded Morgan-Stone lattice $\mathfrak{A}_{01}$ with $(\mathfrak{A}_{01}{\restriction}\Sigma^-_+) \triangleq \mathfrak{A}$ and $(\bot/\top)^{\mathfrak{A}_{01}} \triangleq (a/b)$, and so, for all $\mathfrak{C} \in \mathsf{MS}_{01} \triangleq \{\mathfrak{B}_{01} \mid \mathfrak{B} \in \mathsf{MS}\}$ and $\mathfrak{D} \in \mathsf{MS}_{-2,01} \triangleq (\mathsf{MS}_{01} \setminus \{\mathfrak{MS}_{2,01}\})$, $((\mathfrak{D}{\restriction}\Sigma^-_+) \preceq (\mathfrak{C}{\restriction}\Sigma^-_+)) \Rightarrow (\mathfrak{D} \preceq \mathfrak{C})$. Then, since $\mathfrak{MS}_{2,01} \notin \mathsf{MSA} \supseteq (\mathbf{IS}\mathfrak{MS}_{6,01}) \supseteq \mathsf{MS}_{-2,01}$, while surjective lattice homomorphisms preserve lattice bounds (if any), whereas expansions by constants alone preserve congruences, by (2.8), (2.9) and Theorem 4.4, we immediately get:

Corollary 4.5. *Let* $\mathsf{K} \triangleq (\varnothing|\{\mathfrak{MS}_{2,01}\}$. *Then,* $\mathsf{V} \triangleq (\mathsf{BMSL}|\mathsf{MSA})$ *is the [pre-/quasi-]variety generated by* $\{\mathfrak{MS}_{6,01}, \mathfrak{MS}_{2,01}\} \setminus \mathsf{K}$ *with* $\mathrm{SI}(\mathsf{V}) = \mathbf{I}(\mathsf{MS}_{01} \setminus \mathsf{K})$ *and REDPC scheme* $\mathcal{U}_\Omega^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}$.

This subsumes [3] and also yields a uniform insight into REDPC for Stone and De Morgan algebras, originally given by separate distinct schemes in [13, 22] and a bit enhanced in Corollary 4.7.

4.2. **The lattice of sub-varieties.** [Bounded/] Morgan-Stone lattices[/algebras], satisfying either of the following equivalent — in view of (4.2) — Σ^-_+-identities:

$$(\neg\neg x_0(\vee\neg x_0)) \approx \| \lessapprox (x_0(\vee\neg x_0)), \tag{4.9}$$

are called *[bounded/] (nearly) {De} Morgan lattices[/algebraas]*, their variety being denoted by [B/](N){D}M(L[/A]). Likewise, those, satisfying the Σ^-_+-identity:

$$(x_0 \wedge \neg x_0) \lessapprox x_1, \tag{4.10}$$

are nothing but *[bounded/] Stone lattices[/algebras]* [cf., e.g., [8]], their variety being denoted by [B/]S(L[/A]). Then, members of $[[\mathsf{B}/]\mathsf{B}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}]\mathsf{S}(\mathsf{L}[/\mathsf{A}]))$ are exactly *[bounded/] Boolean lattices[/algebras]*. Further, [bounded/] Morgan-Stone lattices[/algebras], satisfying "either of the former"|"the latter" of the following Σ^-_+-identities:

$$(\neg\neg x_0 \wedge \neg x_0) \approx \| \lessapprox (x_0 \wedge \neg x_0), \tag{4.11}$$

$$\neg\neg x_0 \lessapprox (x_0 \vee (\neg\neg x_1 \vee \neg x_1)), \tag{4.12}$$

"in which case they satisfy the $\Sigma^-_{+[,01]}$-quasi-identities [(4.8) and]:

$$\{(\neg x_0\{\wedge x_1\}) \lessapprox (\neg\neg x_0\{\vee x_2\})(, (\neg x_0 \vee \neg\neg x_0) \approx (\neg x_0 \vee x_0))\} \rightarrow ((\neg x_0\{\wedge x_1\}) \lessapprox ((x_0\{\vee x_2\})), \tag{4.13}$$

[in view of (4.7)]"| are said to be *quasi-|pseudo-strong*, their variety being denoted by [B/](Q|P)SMS(L[/A]). Then, members of

$$[\mathsf{B}/]\mathsf{SMS}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\mathsf{QSMS}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}/]\mathsf{PSMS}(\mathsf{L}[/\mathsf{A}])) \supseteq ([\mathsf{B}/]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cup [\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}]))$$

are said to be *strong*. Next, *[bounded/] ((⌊⌈quasi-|pseudo-⌉strong⌋) {weakly} Kleene⟨-Morgan⟩(-Stone) lattices [/algebras]* are [bounded/] (⌊⌈quasi-|pseudo-⌉strong⌋) De-Morgan(-Stone) lattices[/algebras] satisfying the following Σ^-_+-identity:

$$\mathcal{K}^{\{\mathrm{W}\}}_{\langle \mathrm{M}\rangle} \triangleq ((\langle\neg\neg x_2\wedge\rangle(x_0 \wedge \neg x_0)) \lessapprox (\langle x_2\vee\rangle(\neg x_1 \vee \{\neg\neg\} x_1))),$$

their variety being denoted by

$$[\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\{\mathsf{W}\}\mathsf{K}\langle\mathsf{M}\rangle(\mathsf{S})(\mathsf{L}[/\mathsf{A}]) \supseteq (\varnothing(\cup([\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}])))\ \{\cup[\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\mathsf{K}(\langle\mathsf{M}\rangle\mathsf{S})(\mathsf{L}[/\mathsf{A}])\}$$

$$(\langle[\mathsf{B}/]\mathsf{DM}(\mathsf{L}[/\mathsf{A}]) \cup [\mathsf{B}/](\lfloor\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rfloor)\{\mathsf{W}\}\mathsf{K}(\mathsf{S})(\mathsf{L}[/\mathsf{A}])))$$

{in view of (4.2)}. Likewise, members of

$$[\mathsf{B}/]\mathsf{NK}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}]) \cap [\mathsf{B}/]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}]))$$

are called *[bounded/] nearly Kleene lattices[/algebras]*. Further, the variety of totally negatively-idempotent [bounded] Morgan-Stone lattices, being relatively axiomatized by the Σ^-_+-identity:

$$\neg\neg x_0 \approx \neg x_0, \tag{4.14}$$

is denoted by $[\mathsf{B}]\mathsf{TNIMSL}$. Likewise, the variety of one-element [bounded/] Morgan-Stone lattices[/algebras], being (relatively) axiomatized by the Σ^-_+-identity:

$$x_0 \approx x_1, \tag{4.15}$$

is denoted by $[\mathsf{B}/]\mathsf{OMS}(\mathsf{L}[/\mathsf{A}])$. Furthermore, members of $[\mathsf{B}/](\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])$, satisfying the following Σ^-_+-identity:

$$((\neg x_0 \wedge \neg\neg x_0) \wedge \neg\neg x_1) \lessapprox ((\neg x_0 \wedge x_0) \vee \neg x_1), \tag{4.16}$$

are said to be *almost quasi-strong*, their variety being denoted by

$$[\mathsf{B}/]\mathsf{AQS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \supseteq ([\mathsf{B}/]\mathsf{QS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing])).$$

Then, members of

$$[\mathsf{B}/]\mathsf{AS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}/]\mathsf{AQS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cap$$
$$[\mathsf{B}/]\mathsf{PS}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])) \supseteq ([\mathsf{B}/]\mathsf{S}(\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}]) \cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing]))$$

are said to be *almost strong*, in which case, due to the truth of the Σ_+-quasi-identity

$$\{((x_0 \wedge x_2)(\wedge x_3)) \lessapprox ((x_1 \wedge x_2)(\vee x_4)), (x_0 \vee x_2) \lessapprox (x_1 \vee x_2)\} \to ((x_0(\wedge x_3)) \lessapprox (x_1(\vee x_4))) \tag{4.17}$$

in distributive lattices:

$$([\mathsf{B}](\mathsf{A})\{\mathsf{Q}\}\mathsf{SMSL} \cap [\mathsf{B}]\mathsf{NDML}) = [\mathsf{B}](\mathsf{A})\mathsf{DML}. \tag{4.18}$$

Likewise, members of $[\mathsf{B}/](\mathsf{M}|\{\mathsf{W}\}\mathsf{K})\mathsf{S}(\mathsf{L}[/\mathsf{A}])$, satisfying the following Σ^-_+-identity:

$$(\neg\neg x_0 \wedge \neg\neg x_1) \lessapprox (x_0 \vee \neg x_1), \tag{4.19}$$

are called *[bounded/] almost "De Morgan"|"{weakly} Kleene" lattices[/algebras]*, their variety being denoted by $[\mathsf{B}/]\mathsf{A}(\mathsf{DM}|\{\mathsf{W}\}\mathsf{K})(\mathsf{L}[/\mathsf{A}]) \supseteq ([\mathsf{B}/](\mathsf{DM}|\{\mathsf{W}\}\mathsf{K})(\mathsf{L}[/\mathsf{A}])$ $\cup ([\mathsf{B}]\mathsf{TNIMSL}[/\varnothing]))$. Finally, [bounded/] Morgan-Stone lattices[/algebras], satisfying the optional|non-optional version of the following Σ^-_+-identity:

$$(\neg x_0 \vee \lceil\neg\neg\rceil x_0) \gtrapprox x_1, \tag{4.20}$$

are called *[bounded/] almost Stone|Boolean lattices[/algebras]*, their variety being denoted by $[\mathsf{B}/]\mathsf{A}(\mathsf{S}|\mathsf{B})(\mathsf{L}[/\mathsf{A}])$.

Let[2]

$$\mathcal{MS}_{[01]}\lfloor(\mathfrak{A})\rfloor \triangleq (\{[(4.8),](4.9),((4.9)),(4.10),(4.11),(4.12),\mathcal{K},\mathcal{K}^{\mathrm{W}},$$
$$\mathcal{K}_{\mathrm{M}},\mathcal{K}^{\mathrm{W}}_{\mathrm{M}},(4.16),(4.19),(4.20),\lceil(4.20)\rceil,(4.14)\}\lfloor\cap\mathcal{E}(\mathfrak{A})\rfloor)$$

$\lfloor$where $\mathfrak{A} \in \mathsf{MS}_{[01]}\rfloor$.

[2]From now on, to unify equation environment references, those <not> incorporated into option brackets mean corresponding <non->optional versions of referred quasi-identities.

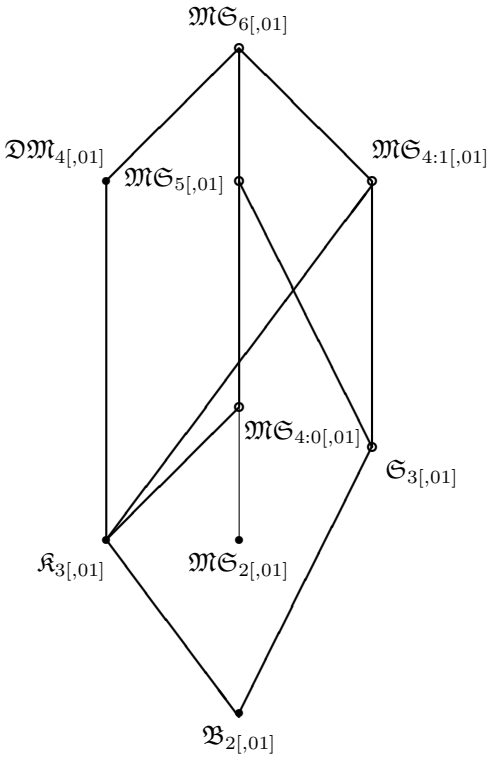

FIGURE 2. The poset $\langle \mathsf{MS}_{[01]}, \preceq \rangle$ [with merely thick lines].

Lemma 4.6. *For any $\mathfrak{A} \in \mathsf{MS}_{[01]}$, $\mathcal{MS}_{[01]}(\mathfrak{A})$ is given by Table 1, in which case the poset $\langle \mathsf{MS}_{[01]}, \preceq \rangle$ is given by Figure 2 with (non-)simple/$\mathfrak{V}^{\langle x_0, \neg x_0 \lceil, \neg\neg x_0 \rceil \rangle}_{\{x_0 \lceil, \neg x_0 \lfloor, \neg\neg x_0 \rfloor \rceil\}}$-implicative members marking (non-)solid circles-nodes [and merely thick lines], and so, for any $\mathfrak{B} \in \mathsf{MS}_{[01]}$, $(\mathsf{MS}_{[01]} \cap \mathbf{H}\mathfrak{B}) \subseteq \mathbf{IS}\mathfrak{B}$. In particular, relative sub-varieties of $\mathsf{MS}_{[01]}$ are exactly its relatively both abstract and hereditary subclasses.*

Proof. Clearly, for any line of Table 1, the identities of the second column of it are true in the algebra of the first one. Conversely,

$$\begin{aligned}
\mathfrak{MS}_{(5|6)[,01]} &\not\models \mathcal{K}^{|\mathrm{W}}_{\|\mathrm{M}}[x_i/\langle 1 - \min(1,i), 1 | \max(1-i, i-1), \min(1,i)\rangle]_{i\in(2\|3)},\\
\mathfrak{S}_{3[,01]} &\not\models ((((4.9))\|(4.9))|((4.19)\|(4.20)))[x_i/(1+i)]_{i\in(1|2)},\\
\mathfrak{DM}_{(4[,01]} &\not\models \mathcal{K}^{\{\mathrm{W}\}}[x_i/(\langle i, i, 1-i\rangle]_{i\in 2},\\
\mathfrak{MS}_{4:1[,01]} &\not\models (4.12)[x_0/\langle 0,1,1\rangle, x_1/\langle 0,0,1\rangle],
\end{aligned}$$

TABLE 1. Identities of $\mathcal{MS}_{[01]}$ true in members of $\mathsf{MS}_{[01]}$.

$\mathfrak{MS}_{6[,01]}$	$\emptyset[\cup\{(4.8)\}]$
$\mathfrak{MS}_{5[,01]}$	$\{[(4.8),]((4.12), \mathcal{K}^{\mathrm{W}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}\}$
$\mathfrak{MS}_{4:0[,01]}$	$\{[(4.8),]((4.9)), (4.12), \mathcal{K}, \mathcal{K}^{\mathrm{W}}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}\}$
$\mathfrak{MS}_{4:1[,01]}$	$\{[(4.8),]((4.11), \mathcal{K}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}_{\mathrm{M}}, \mathcal{K}^{\mathrm{W}}_{\mathrm{M}}, (4.16)\}$
$\mathfrak{DM}_{4[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{K}, \mathcal{K}^{\mathrm{W}}, (4.10), (4.20), \lceil(4.20)\rceil, (4.14)\}$
$\mathfrak{MS}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{[(4.8),]((4.9), (4.11), (4.10)\}$
$\mathfrak{K}_{3[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.10), (4.20), \lceil(4.20)\rceil, (4.14)\}$
$\mathfrak{S}_{3[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.9), ((4.9)), (4.19), (4.20), (4.14)\}$
$\mathfrak{B}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.14)\}$

$$\begin{aligned}\mathfrak{MS}_{4:0[,01]} &\not\models (4.16)[x_i/\langle i,1,i\rangle]_{i\in 2},\\ \mathfrak{K}_{3[,01]} &\not\models ((4.10)|(\lceil(4.20)\rceil\|(4.20)))[x_0/1,x_1/(0|2)],\\ (\mathfrak{B}|\mathfrak{MS})_{2[,01]} &\not\models (4.14|(4.9\|4.11))[x_0/(0|\langle 0,1,0\rangle)][,\\ \mathfrak{MS}_{2,01} &\not\models (4.8)].\end{aligned}$$

Moreover, by Remark 4.2, $\mathfrak{V}_\Omega^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}$ is an REDPC scheme for $[\mathsf{B}]\mathsf{MSL}\supseteq\mathsf{MS}_{[01]}$, in which case, by Corollary 3.4, any simple member $\mathfrak{A}$ of it is $\mathfrak{V}_\Omega^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}$-implicative, and so all those members of MS, which are embeddable into $\mathfrak{A}$, being then $\mathfrak{V}_\Omega^{\langle x_0,\neg x_0,\neg\neg x_0\rangle}$-implicative as well, are simple too. On the other hand,

$$\chi_3^{3\backslash 1}=(\epsilon_3^5\circ\pi_2)\in\hom^{\mathrm{S}}(\mathfrak{S}_{3[,01]},\mathfrak{B}_{2[,01]}),\tag{4.21}$$

in which case $(\ker\chi_3^{3\backslash 1})\in(\mathrm{Co}(\mathfrak{S}_{3[,01]})\setminus\{\Delta_3,3^2\})$, and so $\mathfrak{S}_{3[,01]}$ is not simple. Likewise,

$$\hbar_0\triangleq\{\langle\bar a,[\frac{a_0+a_1+a_2+1}{2}]\rangle\mid\bar a\in MS_{4:0}\}\in\hom^{\mathrm{S}}(\mathfrak{MS}_{4:0[,01]},\mathfrak{K}_{3[,01]}),\tag{4.22}$$

in which case $(\ker\hbar_0)\in(\mathrm{Co}(\mathfrak{MS}_{4:0[,01]})\backslash\{\Delta_{MS_{4:0}},MS_{4:0}^2\})$, and so $\mathfrak{MS}_{4:0[,01]}$ is not simple. Thus, the fact that varieties are abstract, image-closed and hereditary, the simplicity of two-element algebras, the equality (4.11) = $((4.10)[x_0/\neg x_0,x_1/(x_0\wedge\neg x_0)]$, Lemma 3.17, Theorem 4.4, Corollary 4.5, Remarks 4.2, 4.3 and the truth of the identity $(4.9)|(\neg x_0\approx\neg x_1)$ in $(\mathfrak{DM}|\mathfrak{MS})_{4|2}$ complete the argument. □

Corollary 4.7. *Sub-varieties of* $[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$ *form the non-chain distributive lattice with* $29[(+11)/(-9)]$ *elements, whose Hasse diagram with [both thick and/] thin lines is depicted at Figure 3, any (non-)solid circle-node of it being marked by a (non-)semi-simple|filtral|*$\langle\mathfrak{V}_{\{x_0\lceil,\neg x_0\lfloor,\neg\neg x_0\rfloor\rceil\}}^{\langle x_0,\neg x_0\lceil,\neg\neg x_0\rceil\rangle}-\rangle$*implicative variety* $\mathsf{V}\subseteq[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$*, numbered from* $1[+(0/20)]$ *to* $29[+11]$ *according to Table 2 with* $\Bbbk\triangleq(9\cdot(1[/0]))$ *[as well as* $\ell\triangleq(29\cdot(0/1))$*/ and* $\mathsf{MS}_{\mathsf{V}[,01]}\triangleq\max_{\preceq}((\mathsf{MS}_{[-2,01]}[\cup\mathsf{K}])\cap\mathsf{V})$*,*

Table 2. Maximal subdirectly-irreducibles of varieties of [bounded/] Morgan-Stone lattices[/algebras].

$1[+\ell]$	$[\mathsf{B}]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{6[,01]}\}[\cup\mathsf{K}]$
$2[+\ell]$	$[\mathsf{B}]\mathsf{PS}\langle\mathsf{WK}\rangle\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$3\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{WK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{MS}_{4:1[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$5[+\ell]$	$[\mathsf{B}]\mathsf{PSWKS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]}\}[\cup\mathsf{K}]$
$6\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{K}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:i[,01]}\mid i\in 2\}\lceil\cup\{\mathfrak{DM}_{4[,01]}\}\rceil[\cup\mathsf{K}]$
$8\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{PSK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{S}_{3[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$10[+\ell]$	$[\mathsf{B}]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$11[+\ell]$	$[\mathsf{B}]\mathsf{NK}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]}\}[\cup\mathsf{K}]$
12	$[\mathsf{B}]\mathsf{TNIMSL}$	$\{\mathfrak{MS}_{2[,01]}\}$
$22\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QSMS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$23\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QS}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$24\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{SMS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$25\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{DM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$26\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{K}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$27\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\{\mathsf{W}\}\mathsf{K}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{K}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$28\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$29\lfloor-\Bbbk\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{B}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{B}_{2[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
21	$[\mathsf{B}/]\mathsf{OMS}(\mathsf{L}[/\mathsf{A}])$	$\varnothing$

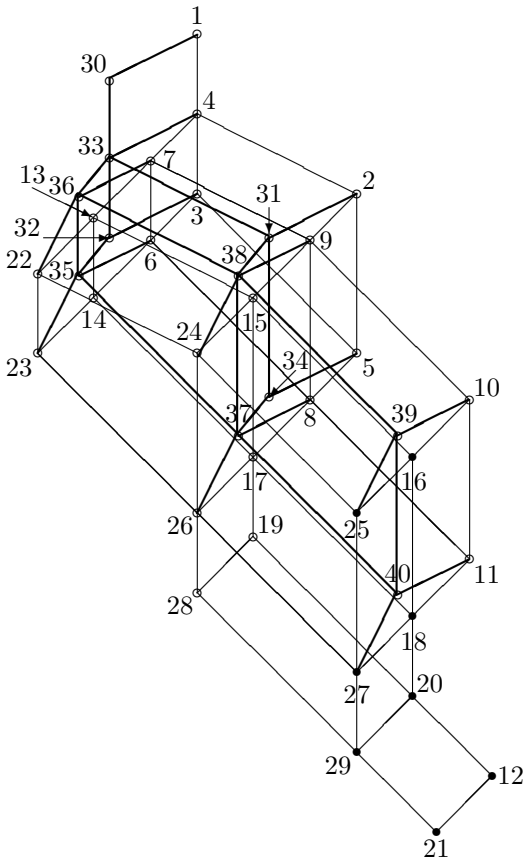

FIGURE 3. The lattice of varieties of [bounded/] Morgan-Stone lattices[/algebras].

where $\mathsf{K} \triangleq (\{\mathfrak{MS}_{2[,01]}\}[/\varnothing])$*, given by the third column, in which case* $\mathrm{SI}(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{MS}_{\mathsf{V}[,01]}$*, and so* V *is the (pre-‖quasi-)variety generated by* $\mathsf{MS}_{\mathsf{V}[,01]}$*, while* [B]SMSL *is that generated by* $\{\mathrm{SI}\}([\mathsf{B}]\mathsf{DML} \cup [\mathsf{B}]\mathsf{SL})$ *with REDPC scheme* $\mho^{\langle x_0, \neg x_0\rangle}_{\{x_0, \neg x_0\}}$*, whereas any disjunctive sub-pre-variety of* [B/]MS(L[/A]) *is equational, and so is any quasi-equational//finitely implicative one.*

Proof. We use Lemma 4.6 tacitly. Then, the intersections of $\mathsf{MS}_{[-2,01]}[\cup\mathsf{K}]$ with the 29[(+11)/(−9)] sub-varieties of [B/]MS(L[/A]) involved are exactly all lower cones of the poset $\langle \mathsf{MS}_{[-2,01]}[\cup\mathsf{K}], \preceq\rangle$, i.e., the sets appearing in the third column of Table 2 are exactly all anti-chains of the poset. So, (2.8), (2.9), (4.1), (4.5), Theorems 3.7, 3.13, 4.4, Corollaries 3.14, 4.5, Lemmas 3.17, 3.30, [21, Remark 2.4], the truth of the Σ^-_+-quasi-identities in $\{(\bigcup_{i\in 2}\{(x_2 \wedge x_i) \lessapprox (x_{1-i} \vee x_3), (x_2 \wedge \neg x_i) \lessapprox (\neg x_{1-i} \vee x_3)\}) \to ((x_2 \wedge \neg\neg x_j) \lessapprox (\neg\neg x_{1-j} \vee x_3)) \mid j \in 2\}$ in $\{\mathfrak{DM}_4, \mathfrak{S}_3\}$ and the fact that pre-varieties are abstract and hereditary complete the argument. □

It is in this sense that [B]SMSL is the implicational/[quasi-]equational join of [B]DML and [B]SL. Likewise, QSMSL is the greatest sub-variety of MSL not containing $\mathfrak{MS}_2$, in which case it is that containing the Σ^-_+-reduct of no member of BMSL \ MSA, and so it is in this sense that it is viewed as "an equational unbounded approximation of MSA" due to absence of any class of Σ^-_+-implications axiomatizing MSA relatively to BMSL, simply because any sub-pre-variety of MSL including $\mathsf{K}' \triangleq (\mathsf{MS} \setminus \{\mathfrak{MS}_2\})$ contains $\mathfrak{MS}_2 \in \mathbf{S}\mathsf{K}'$ (this is why the node 30 at Figure 3

corresponds to no sub-variety of MSL). The finite lattice of its sub-quasi-varieties is found in the next Section. This task (as well as that solved in [18]) cannot be solved with using tools elaborated in [21] because of Proposition 5.11 therein. And what is more, despite of implicativity of {sub-varieties of} $[\mathsf{B}](\mathsf{A})\mathsf{DML}$ and Remark 3.1, we have:

Remark 4.8. Clearly, $\theta \triangleq (\Delta_3 \cup (\{1\} \times 3)) \subseteq (3^2 \setminus (\{0,2\}^2 \setminus \Delta_{\{0,2\}}))$ forms a subalgebra of $\mathfrak{K}^2_{3[,01]}$, in which case, if $\mathfrak{K}_{3[,01]}$ had a dual discriminator δ, then we would have $2 = \delta^{\mathfrak{K}_{3[,01]}}(1,0,2)\ \theta\ \delta^{\mathfrak{K}_{3[,01]}}(0,0,2) = 0$, and so, by Theorem 4.4 and Corollary 2.7, no sub-variety of $[\mathsf{B}]\mathsf{MSL}$ containing $|$"the non-simple subdirectly-irreducible" $\mathfrak{K}|\mathfrak{S}_{3[,01]}$ (viz., including $[\mathsf{B}](\mathsf{K}|\mathsf{S})\mathsf{L}$; cf. Corollary 4.7) is {dual} discriminator. □

On the other hand, the majority term μ_+ for the variety of lattices, being a dual discriminator for $\mathfrak{D}_2$, is that for $\{\mathfrak{B}_{2[,01]}, \mathfrak{MS}_{2[,01]}\}$, in which case, by Corollary 4.7, sub-varieties of $[\mathsf{B}]\mathsf{ABL}$ are dual μ_+-discriminator, and so, by Remark 4.8, these are exactly all dual (μ_+-)discriminator sub-varieties of $[\mathsf{B}]\mathsf{MSL}$. Nevertheless, since $\neg x_0 \approx \top$ is true in $\mathfrak{MS}_{2,01}$, its isomorphic copy by $\pi_0{\restriction}MS_2$ is term-wise-definitionally equivalent to $\mathfrak{D}_{2,01}$ generating the variety BDL (cf., e.g., [2] or Lemma 4.1), in its turn, being well-known (e.g., due to [5] {cf. [21, Lemma 2.10]} and existence of a three-element subdirect square of $\mathfrak{D}_{2,01}$ with carrier $2^2 \setminus \{\langle 0,1\rangle\}$, though $3 \neq 1$ is odd), in which case $\mathfrak{MS}_{2[,01]}$ has no congruence-permutation term, for, otherwise, $\mathfrak{D}_{2,01}$ would have one, and so, by Corollaries 2.7 and 4.7, $[\mathsf{B}]\mathsf{BL}$ is the only discriminator sub-variety of $[\mathsf{B}]\mathsf{MSL}$.

5. Quasi-varieties of almost quasi-strong Morgan-Stone lattices

5.1. **Non-idempotencity versus two-valued Boolean homomorphisms.** Given any $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$, $(\mathsf{N})\mathsf{IK}$ stands for the class of (non-)idempotent members of K (in which case it is the relative sub-quasi-variety of K, relatively axiomatized by the Σ^-_+-quasi-identity:

$$(\neg x_0 \approx x_0) \to (x_0 \approx x_1), \tag{5.1}$$

and so a quasi-variety, whenever K is so).

Given any $\mathsf{K}', \mathsf{K}'' \subseteq [\mathsf{B}]\mathsf{MSL}$, set $(\mathsf{K}' \otimes \mathsf{K}'') \triangleq \{\mathfrak{A} \times \mathfrak{B} \mid (\mathfrak{A}|\mathfrak{B}) \in (\mathsf{K}'|\mathsf{K}'')\}$.

Let $\boldsymbol{\mu} \triangleq (\neg x_0 \vee \neg\neg x_0) \in \mathrm{Tm}^1_{\Sigma^-_+}$ and $\boldsymbol{\pi} \triangleq ((x_0 \vee \neg x_1) \wedge x_1) \in \mathrm{Tm}^2_{\Sigma^-_+}$. Then, given any $\tau \in \mathrm{Tm}^1_{\Sigma^-_+}$, [by induction on any $i \in \omega$] put

$$\boldsymbol{\iota}_{\tau,1[+i+1]} \triangleq ((x_0[[x_0/\boldsymbol{\pi}]])[x_0/(\tau[x_0/x_{0[+i+1]}])[,x_1/\boldsymbol{\iota}_{\tau,i+1}]]) \in \mathrm{Tm}^{1[+i+1]}_{\Sigma^-_+}.$$

Finally, for any $n \in (\omega \setminus 1)$, set $\boldsymbol{\iota}_n \triangleq \boldsymbol{\iota}_{\boldsymbol{\mu},n} \in \mathrm{Tm}^n_{\Sigma^-_+}$.

Lemma 5.1. *Any (non-one-element finitely-generated)* $\mathfrak{A} \in [\mathsf{B}]\mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, *in which case* $\mathsf{I}[\mathsf{B}]\mathsf{SMSL} \subseteq [\mathsf{B}]\mathsf{DML}$, *and so* $[\mathsf{B}]\mathsf{S}(\mathsf{M}|\mathsf{K})\mathsf{SL} = (\mathsf{NI}[\mathsf{B}]\mathsf{S}(\mathsf{M}|\mathsf{K})\mathsf{SL} \cup [\mathsf{B}](\mathsf{M}|\mathsf{K})\mathsf{L})$. *In particular,* $(\mathsf{NI}[\mathsf{B}]\mathsf{SMSL} \cup [\mathsf{B}]\mathsf{KL}) = (\mathsf{NI}[\mathsf{B}]\mathsf{SMSL} \cup [\mathsf{B}]\mathsf{SKSL})$, *while* $\mathsf{NIMS}_{[01]} = \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}$, *whereas any variety* $\mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ *with* $\mathsf{NIV} \not\subseteq [\mathsf{B}]\mathsf{OMSL}$ *contains* $\mathfrak{B}_{2[,01]}$.

Proof. The "if" part is by the fact that $\mathfrak{B}_{2[,01]}$ has no idempotent element. (Conversely, assume $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) = \varnothing$, in which case, by (4.21), $\hom(\mathfrak{A}, \mathfrak{S}_{3[,01]}) = \varnothing$, and so $(\hom(\mathfrak{A}, \{\mathfrak{MS}_{6[,01]}[,\mathfrak{MS}_{2,01}]\}) \cap (\mathrm{img}\,\epsilon^5_3)^A) = \varnothing$. Then, by (2.8), Theorem 4.4 [resp., Corollary 4.5] and the right alternative of the following claim, $\mathfrak{A}$, being non-one-element, is idempotent:)

Claim 5.2. *Let* $\mathfrak{B} \in [\mathsf{B}]\mathsf{MSL}$, $n \in (\omega \setminus 1)$, $\bar{b} \in B^n$, $\mathfrak{C} \in \{\mathfrak{MS}_{6[,01]}[,\mathfrak{MS}_{2,01}]\}$, $h \in (\hom(\mathfrak{B}, \mathfrak{C}) \setminus (\varnothing|(\mathrm{img}\,\epsilon^5_3)^B))$ *and* $\tau \in (\mathrm{Tm}^1_{\Sigma^-_+}|\{\boldsymbol{\mu}\})$. *Suppose "for each* $i \in n$,

$h(\neg^{\mathfrak{B}}\tau^{\mathfrak{B}}(b_i)) \leqslant^{\mathfrak{C}} h(\neg^{\mathfrak{B}}\neg^{\mathfrak{B}}\tau^{\mathfrak{B}}(b_i))$"| "$\mathfrak{B}$ *is generated by* $\operatorname{img}\bar{b}$". *Then,* $h(\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_{\tau,n}(\bar{b})) \leqslant^{\mathfrak{C}} | = h(\neg^{\mathfrak{B}}\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_{\tau,n}(\bar{b}))$*, in which case* $\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_{\tau,n}(\bar{b}) \leqslant^{\mathfrak{B}} \neg^{\mathfrak{B}}\neg^{\mathfrak{B}}\boldsymbol{\iota}^{\mathfrak{B}}_{\iota,n}(\bar{b})$*, and so the* Σ^-_+*-(quasi-)identity* $(\{\neg x_j \lessapprox \neg\neg x_j \mid j \in n\} \rightarrow)(\neg\boldsymbol{\iota}_{(\tau,)n} \lessapprox \neg\neg\boldsymbol{\iota}_{(\tau,)n})$ *of rank* n *is true in* [B]MSL.

Proof. By induction on n, (2.8), Theorem 4.4 [resp., Corollary 4.5], the equality $(C \setminus (\operatorname{img}\epsilon^5_3)) = \Im^{\mathfrak{C}}_{\neg}$ as well as the truth of the Σ^-_+-quasi-identities

$$\varnothing\|\{\neg x_0 \approx \neg\neg x_0\}) \quad \rightarrow \quad (\neg\boldsymbol{\mu} \lessapprox \| \approx \neg\neg\boldsymbol{\mu}), \tag{5.2}$$

$$\{\neg x_j \lessapprox \| \approx \neg\neg x_j, \neg x_{1-j} \lessapprox \neg\neg x_{1-j}\} \quad \rightarrow \quad (\neg\boldsymbol{\pi} \lessapprox \| \approx \neg\neg\boldsymbol{\pi}), \tag{5.3}$$

where $j \in 2$, in [B]MSL, in their turn, being due to that of (4.1), (4.5) and (4.6). □

Finally, (2.8), (4.21), Corollary 4.7 and absence of proper subalgebras of $\mathfrak{B}_{2[,01]}$ complete the argument. □

Lemma 5.3. $\mathfrak{K}_{3[,01]}\restriction(\{0,2\}(\cup\{1\}))$ *is embeddable into any* $\mathfrak{A} \in ([\mathsf{MSA} \cap \mathsf{B}]\mathsf{MSL}) \setminus ((\mathsf{NI}[\mathsf{B}]\mathsf{MSL} \cup [\mathsf{B}]\mathsf{TNIMSL}))[= ((\mathsf{I})\mathsf{MSA} \setminus \mathsf{OMSA})] \supseteq ((\mathsf{I})([\mathsf{MSA} \cap \mathsf{B}]\mathsf{QSMSL}) \setminus [\mathsf{B}]\mathsf{OMSL})$.

Proof. Then, we have some $(a,)b \in A$ such that $(\neg^{\mathfrak{A}}a =)((a\vee^{\mathfrak{A}})((a\wedge^{\mathfrak{A}})\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b)) \neq \neg^{\mathfrak{A}}b$, in which case, by (4.1), (4.3), (4.5) and (4.6) [as well as (4.7) and (4.8)], $c \triangleq (([\perp^{\mathfrak{A}}\wedge^{\mathfrak{A}}](a\wedge^{\mathfrak{A}})(\neg^{\mathfrak{A}}b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b))) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c \leqslant^{\mathfrak{A}} (a \leqslant^{\mathfrak{A}})d \triangleq \neg^{\mathfrak{A}}c$, while $\neg^{\mathfrak{A}}d = c(\neq a$, for, otherwise, we would have $\neg^{\mathfrak{A}}b \geqslant^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b$, the latter implying, by (4.3) and (4.6), $\neg^{\mathfrak{A}}b \leqslant^{\mathfrak{A}} a)$, whereas $d \notin \{c(,a)\}$, for, otherwise, we would get $((c\vee^{\mathfrak{A}})((c\wedge^{\mathfrak{A}})([\perp^{\mathfrak{A}}\wedge^{\mathfrak{A}}]\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b))) = ((a\vee^{\mathfrak{A}})((a\wedge^{\mathfrak{A}})([\top^{\mathfrak{A}}\vee^{\mathfrak{A}}]\neg^{\mathfrak{A}}b)))$, and so $\{\langle 0,c\rangle, (\langle 1,a\rangle,)\langle 2,d\rangle\}$ is an embedding of $\mathfrak{K}_{3[,01]}\restriction(\{0,2\}(\cup\{1\}))$ into $\mathfrak{A}$, as required, for, by Corollary 4.7, $([\mathsf{B}]\mathsf{QSMSL} \cap [\mathsf{B}]\mathsf{TNIMSL}) = [\mathsf{B}]\mathsf{OMSL} \subseteq \mathsf{NI}[\mathsf{B}]\mathsf{MSL}$. □

The stipulation of quasi-strength [resp., MS-algebraicity] here can be neither omitted nor replaced by the one of pseudo-strength nor, even, weakened with replacing it by that of almost quasi-strength, when taking $\mathfrak{A} = \mathfrak{MS}_{2[,01]}$.

The above two lemmas, by (2.1), (2.7) with $I = 2$, (2.8), (2.10), Corollary 4.7, the locality of quasi-varieties, the quasi-equationality of finitely-generated pre-varieties, the simplicity of two-element algebras and the equality $\mathsf{NI}[\mathsf{B}]\mathsf{TNIMSL} = [\mathsf{B}]\mathsf{OMSL}$, immediately yield:

Corollary 5.4. *Let* $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$ *and* $\mathsf{P} \triangleq \mathbf{PV}(\mathsf{K})$*. Suppose either* $\mathfrak{B}_{2[,01]} \in | \preceq (\mathsf{P}|\mathsf{K})$ *(more specifically, either* $[\mathsf{B}]\mathsf{OMSL} \not\supseteq (\mathsf{K}\|\mathsf{P}) \subseteq [\mathsf{B}]\mathsf{QSMSL}$ *or both* $[\mathsf{B}]\mathsf{OMSL} \not\supseteq \mathsf{NI}(\mathsf{K}\|\mathsf{P})$ *and* P *is equational) or* $\mathsf{IK} = \varnothing$*. Then,* $\mathsf{NIP} = \mathbf{PV}((\mathsf{IK} \otimes (\{\mathfrak{B}_{2[,01]}\} \cap (\mathsf{P}|(\mathbf{IS}\mathsf{K}))))\cup\mathsf{NIK})$*, in which case, for any variety* $\mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ {*such that* $[\mathsf{B}]\mathsf{BL} \subseteq \mathsf{V}$ *(i.e.,* $[\mathsf{B}]\mathsf{TNIMMSL} \not\supseteq \mathsf{V}$*)*}*,* $\mathsf{NIV} = (\mathbf{P}//\mathbf{Q})\mathbf{V}(\varnothing\{\cup((\mathsf{MS}_{\mathsf{V}[,01]} \setminus \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}) \otimes \{\mathfrak{B}_{2[,01]}\})\}\cup(\mathsf{MS}_{\mathsf{V}[,01]} \cap \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}))$*, and so* $\mathsf{NI}[\mathsf{B}/]\{(\mathsf{PSM})|(\mathsf{WK}\langle\mathsf{M}\rangle)|(\mathsf{PSWK})\}\mathsf{S}(\mathsf{L}[/\mathsf{A}])$ *is the pre-//quasi-variety generated by* $(\{\mathfrak{MS}_{(6\{-1\})[,01]}\}\{\cup(\{\mathfrak{DM}_{4[,01]}\}|\{\mathfrak{MS}_{(4:1)[,01]}\langle,\mathfrak{DM}_{4[,01]}\rangle\}|\varnothing)\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])\otimes\{\mathfrak{B}_{2[,01]}\}$*, while* $\mathsf{NI}[\mathsf{B}/]\{(\mathsf{PS})|(\langle\mathsf{A}\rangle\lfloor\mathsf{Q}\rfloor\mathsf{S})\}(\{|\mathsf{M}\|\}(\mathsf{K}\lceil\mathsf{M}\rceil))\mathsf{S}(\mathsf{L}[/\mathsf{A}])$ *is the one generated by* $((\{\mathfrak{MS}_{(4:i)[,01]} \mid i \in (2\{\setminus(\{1\}|(2\lfloor\cap 1\rfloor))\})\} \cup (\{|\{\mathfrak{DM}_{4[,01]}\}\|\}(\varnothing \cup (\varnothing|(\varnothing \cup (\{\mathfrak{K}_{3[,01]}\}\lceil\cap\varnothing\rceil)))\lceil\cup\{\mathfrak{DM}_{4[,01]}\}\rceil)) \cup ((\varnothing[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])|(\varnothing\langle\cup(\{\mathfrak{MS}_{2[,01]}\}[/\varnothing])\rangle))\otimes\{\mathfrak{B}_{2[,01]}\})\cup(\{\mathfrak{S}_{3[,01]}\}|(\{\mathfrak{S}_{3[,01]}\}\lfloor\setminus\{\mathfrak{S}_{3[,01]}\}\rfloor))$*, whereas* $\mathsf{NI}[\mathsf{B}/]\{\mathsf{N}\}(\mathsf{M}|\mathsf{K})(\mathsf{L}[/\mathsf{A}])$ *is that generated by* $((\{((\mathfrak{DM})|\mathfrak{K})_{(4|3)[,01]}\}\{\setminus(\varnothing|\{\mathfrak{K}_{3[,01]}\})\})\{\cup\{\mathfrak{MS}_{(4:0)[,01]}\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)]\})\otimes\{\mathfrak{B}_{2[,01]}\}$*. In particular, any (non-one-element)* $\mathfrak{A} \in [\mathsf{B}]\mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$.

Corollary 5.5. $\mathsf{NI}[\mathsf{B}]\mathsf{MSL} \cup [\mathsf{B}]\mathsf{TNIMSL}$ *is the sub-quasi-variety of* $[\mathsf{B}]\mathsf{MSL}$ *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0 \approx x_0) \rightarrow (x_0 \approx \neg x_1) \tag{5.4}$$

and is the pre-/quasi-variety generated by $\{\mathfrak{MS}_{6[,01]} \times \mathfrak{B}_{2[,01]}, \mathfrak{MS}_{2[,01]}\}$.

Proof. Clearly, (5.4) = (5.1$[x_1/\neg x_1]$) is true in both $\mathsf{NI[B]MSL}$ and $\mathfrak{MS}_{2[,01]}$. Conversely, any $\mathfrak{A} \in \mathsf{I[B]MSL}$, satisfying (5.4), has an idempotent element a, in which case, for any $b \in A$, as $\mathfrak{A} \models (5.4)[x_0/a, x_1/(\neg^{\mathfrak{A}})b]$, we have $\neg^{\mathfrak{A}} b = a (= \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b)$, and so $\mathfrak{A} \in \mathsf{[B]TNIMSL}$. Then, Corollaries 4.7 and 5.4 complete the argument. □

Likewise, we have:

Corollary 5.6. *For any (equational) /quasi-equational pre-variety* $\mathsf{P} \subseteq \mathsf{[B]MSL}$*, the class* $\mathsf{NIP} \cup (\mathsf{P} \cap \mathsf{[B]\{W\}KSL})$ *is the relative/ sub-quasi-variety of* P *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0 \approx x_0) \to (x_0 \lessapprox (\{\neg\neg\} x_1 \vee \neg x_1)) \tag{5.5}$$

(and is the pre-|quasi-variety generated by $\mathsf{MS}_{\mathsf{P}\cap\mathsf{[B]\{W\}KSL})[,01]} \cup ((\mathsf{MS}_{\mathsf{V}[,01]} \setminus \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}) \otimes \{\mathfrak{B}_{2[,01]}\})$*). In particular,* $\mathsf{NI[B]}\lfloor\mathsf{A}\rfloor(\mathsf{D}\langle\|\lceil\mathsf{Q}\rceil\mathsf{S}\rangle)\mathsf{M}\langle\mathsf{S}\rangle\mathsf{L} \cup \lfloor\mathsf{A}\rfloor\langle\lceil\mathsf{Q}\rceil\mathsf{S}\rangle\mathsf{K}\langle\mathsf{S}\rangle\mathsf{L}$ *is the sub-quasi-variety of* $\mathsf{[B]}\lfloor\mathsf{A}\rfloor(\mathsf{D}\langle\|\lceil\mathsf{Q}\rceil\mathsf{S}\rangle)\mathsf{M}\langle\mathsf{S}\rangle\mathsf{L}$ *relatively axiomatized by either version of* (5.5) *and is the pre-|quasi-variety generated by*

$$(\{\mathfrak{DM}_{4[,01]}\lfloor, \mathfrak{MS}_{2[,01]}\rfloor\} \otimes \mathfrak{B}_{2[,01]}) \cup (\{\mathfrak{K}_{3[,01]}\langle, \mathfrak{S}_{3[,01]}\rangle\}\langle\lceil\cap\varnothing\rceil\rangle\langle\lceil\cup\{\mathfrak{K}_{4:1[,01]}\}\rceil\rangle.$$

Proof. Clearly, (5.5) is satisfied in $\mathsf{NIP} \cup (\mathsf{P} \cap \mathsf{[B]\{W\}KSL})$. Conversely, consider any $\mathfrak{A} \in \mathsf{IP}$ satisfying (5.5) and any $a, b \in A$, in which case there is some $c \in A$ such that $\neg^{\mathfrak{A}} c = c$, and so, as $\mathfrak{A}(5.5)[x_0/c, x_1/(a|b)]$, we have $c \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}}(a\|b) \vee^{\mathfrak{A}} \{\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\}(a\|b))$. Then, by (4.2), (4.3) and (4.5) {as well as (4.6)}, we get $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} c$, in which case $\mathfrak{A} \in (\mathsf{P} \cap \mathsf{[B]\{W\}KSL})$, and so Corollaries 4.7, 5.3 and 5.4 complete the argument. □

More generally, we, clearly, have:

Lemma 5.7. *For any* $\alpha \in (\infty \setminus 1)$*, any* $\mathfrak{I} \subseteq (\wp(\mathrm{Eq}^{\alpha}_{\Sigma^-_{+[,01]}}) \times \mathrm{Eq}^{\alpha}_{\Sigma^-_{+[,01]}})$ *and any* $\mathsf{K} \subseteq \mathsf{A}_{\Sigma^-_{+[,01]}}$, $(\mathsf{K} \cap (\mathrm{Mod}(\mathfrak{I}) \cup \mathrm{Mod}((5.1))) = (\mathsf{K} \cap \mathrm{Mod}(\{(\{\neg x_\alpha \approx x_\alpha\} \cup \Gamma) \to \Phi \mid (\Gamma \to \Phi) \in \mathfrak{I}\}))$.

This, by Corollaries 4.7, 5.4 and Lemma 4.6, immediately yields:

Corollary 5.8. *Let* $\mathsf{V} \triangleq \mathsf{[B]}\langle\lceil\mathsf{A}\rceil\mathsf{QS}\rangle\mathsf{MSL}$*. Then,*

$$(\mathsf{NIV}\{\langle\cap\mathsf{[B]KSL}\rangle\} \cup (\mathsf{[B](A)}\lfloor\mathsf{QS}\rfloor\mathsf{M}\lfloor\mathsf{S}\rfloor\mathsf{L}\{\cap\mathsf{[B]KSL}\}\langle\cap\mathsf{V}\rangle)$$

is the sub-quasi-variety of V *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$\{\neg x_{1(+1)} \approx x_{1(+1)}\lfloor, x_0 \lessapprox \neg x_0\rfloor\} \to (((\neg\neg x_0(\wedge\neg\neg x_1)) \approx (x_0(\vee\neg x_1))) \tag{5.6}$$

{*collectively with the one* $(\{\neg x_2 \approx x_2\}\langle\cap\varnothing\rangle) \to \mathcal{K}\}$ *<and is the pre-/quasi-variety generated by* $\{\mathfrak{MS}_{4:1[,01]} \times \mathfrak{B}^{\lfloor 0\rfloor}_{2[,01]}\lceil, \mathfrak{MS}_{2[,01]} \times \mathfrak{B}^{(0)}_{2[,01]}\rceil, \mathfrak{DM}^{\{\langle 0\rangle\}}_{4[,01]} \times \mathfrak{B}^{0\{+1\langle -1\rangle\}}_{2[,01]}\{, \mathfrak{K}^{\lfloor 0\rfloor}_{3[,01]}\}\}$>.

5.2. **Regularizations versus regularity.**

Definition 5.9 (cf. [18, Definition 4.6] for the non-otional case)**.** Members of any /quasi-equational $\mathsf{K} \subseteq \mathsf{[B]MSL}$, satisfying the Σ^-_+-quasi-identity of rank $2\{+1\}$:

$$\mathcal{R}^{(\mathrm{W})}_{\{\mathrm{M}\}} \triangleq (((\{\neg x_0 \lessapprox x_0, (x_0 \wedge \neg x_1) \lessapprox ((\neg x_0 \vee x_1)\}) \to ((\neg x_1\{\wedge\neg\neg x_2\}) \lessapprox ((\neg\neg)x_1\{\vee x_2\})))$$

are called *(weakly-)*{*Morgan-*}*regular*, their relative/ sub-quasi-variety of K being denoted by $((\mathsf{W})\{\mathsf{M}\}\mathsf{RK}$. □

Given any [bounded] {quasi-strong} ⟨(weakly) Kleene⟩ MS lattice $\mathfrak{A}$, by (4.1), (4.2), (4.3) and (4.5) (as well as (4.6)) ⟨together with $\mathcal{K}^{(\mathrm{W})}$⟩ {collectively with (4.13)}, $(\mathcal{I}|\mathcal{F})^{\mathfrak{A}}_{(\mathrm{W})} \triangleq \{a \in A \mid (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})a(\leqslant \mid \geqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}a\} \supseteq \{b(\wedge|\vee)^{\mathfrak{A}}\neg^{\mathfrak{A}}b \mid b \in A\} \neq \varnothing$, for $A \neq \varnothing$, is a|an lower|upper cone of the poset $\langle A, \leqslant^{\mathfrak{A}}\rangle$ ⟨being an|a ideal|filter of $\mathfrak{A}\restriction\Sigma_+$⟩ such that $\neg^{\mathfrak{A}}[(\mathcal{I}|\mathcal{F})^{\mathfrak{A}}_{(\mathrm{W})}] \subseteq (\mathcal{F}|\mathcal{I})^{\mathfrak{A}}_{(\mathrm{W})} (= ((\mathcal{F}|\mathcal{I})^{\mathfrak{A}} \cup ((\mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}\{\cap\varnothing\})|\varnothing)))$ ⟨in which case $\mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})} \triangleq ((\mathcal{F}^{\mathfrak{A}}_{(\mathrm{W})} \times \{1\}) \cup (\mathcal{I}^{\mathfrak{A}}_{(\mathrm{W})} \times \{0\}))(\{= \mathfrak{R}^{\mathfrak{A}}\})$ forms a subalgebra of $\mathfrak{A} \times \mathfrak{B}_{2[,01]}$ such that, for every $\bar{d} \in \mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})}$, $(d_1 = 1) \Rightarrow (d_0 \in \mathcal{F}^{\mathfrak{A}}_{(\mathrm{W})})$, and so the *(weak) regularization* $\mathfrak{R}_{(\mathrm{W})}(\mathfrak{A}) \triangleq ((\mathfrak{A} \times \mathfrak{B}_{2[,01]})\restriction\mathfrak{R}^{\mathfrak{A}}_{(\mathrm{W})})(\{= \mathfrak{R}(\mathfrak{A})\})$ *of* $\mathfrak{A}$ is (weakly) regular⟩. Then, $(\pi_0\restriction\mathfrak{R}^{\mathfrak{S}_{3[,01]}}) \in \hom(\mathfrak{R}(\mathfrak{S}_{3[,01]}), \mathfrak{S}_{3[,01]})$ is bijective, so, by Corollary 4.7, $\mathfrak{S}_{3[,01]} \in \mathsf{R[B]SKSL}$. Likewise, $(\epsilon^4_2\|\{\langle i, \langle \chi^{4\setminus 3}_4(i) + \chi^{4\setminus 1}_4(i), \chi^{4\setminus 2}_4(i)\rangle\rangle \mid i \in 4\}) \in \hom((\mathfrak{B}\|\mathfrak{K})_{(2\|4)[,01]}, \mathfrak{K}_{4[,01]}\|\mathfrak{R}(\mathfrak{K}_{3[,01]}))$ is injective‖bijective, so, by Corollary 4.7, $(\mathfrak{B}\|\mathfrak{K})_{(2\|4)[,01]} \in \mathsf{R[B]KL}$.

Lemma 5.10. *Any ((weakly) {Morgan-}regular [bounded/] MS lattice[/algebra]* $\mathfrak{A}$ *is a [bounded/] (weakly) Kleene-{Morgan-}Stone lattice[/algebra].*

Proof. Consider any $a, b\{, c\} \in A$. Let $d \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}a)$ and $e \triangleq ((b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b) \wedge^{\mathfrak{A}} d)$, in which case, by (4.5), we have $\neg^{\mathfrak{A}}d = (\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a) \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}a \leqslant^{\mathfrak{A}} d$, and so, by (4.1) and (4.5), we get $(d \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}e) = ((d \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b)) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}d) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}d \vee^{\mathfrak{A}} (b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b)) \wedge^{\mathfrak{A}} d) = (\neg^{\mathfrak{A}}d \vee^{\mathfrak{A}} e)$. Then, since $\mathfrak{A} \models \mathfrak{R}^{(\mathrm{W})}_{\{\mathrm{R}\}}[x_0/d, x_1/e\{, x_2/c\}]$, by (4.1), (4.2) and (4.5) (as well as (4.6)), we eventually get $((a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) = (\neg^{\mathfrak{A}}d\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} (((\neg^{\mathfrak{A}}b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}d)\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) = (\neg^{\mathfrak{A}}e\{\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})e\{\vee^{\mathfrak{A}}c\}) = ((((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b) \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})d)\{\vee^{\mathfrak{A}}c\}) \leqslant^{\mathfrak{A}} (((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b)\{\vee^{\mathfrak{A}}c\})$, as required. □

Corollary 5.11. $(\varnothing\{\cup\{\mathfrak{S}_{3[,01]}\langle, \mathfrak{DM}_{4[,01]}\rangle(, \mathfrak{MS}_{2[,01]})\}\}) \subseteq \mathsf{(W)\langle M\rangle R[B]M\{S\}L} \subseteq \mathsf{(NI[B]K\{\langle M\rangle S\}L(\cup[B]TNIMSL)\langle\cup[B](A)DML\rangle)}$. *In particular,* $\mathsf{[B](A)\langle\{S\}M\rangle\{S\}L} \subseteq [B]\mathsf{(W)\langle M\rangle RSMSL}$.

Proof. The first inclusion is immediate. For proving the second one, consider any $\mathfrak{A} \in \mathsf{(W)\langle M\rangle R[B]M\{S\}L}$ and any $a, b, c\langle, d(, e)\rangle \in A$ such that $\neg^{\mathfrak{A}}a = a$, in which case, as $\mathfrak{A} \models ((4.1)|\mathfrak{R}^{(\mathrm{W})}_{\langle\mathrm{R}\rangle})[x_0/a, x_1/(c|(a \wedge^{\mathfrak{A}} c))\langle|, x_2/d\rangle]$ (and $\mathfrak{A} \models (4.5)[x_0/\neg^{\mathfrak{A}}a, x_1/\neg^{\mathfrak{A}}c]$), we have $(\neg^{\mathfrak{A}}c\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}c)\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) = (\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} c)\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})(a \wedge^{\mathfrak{A}} c)\langle\vee^{\mathfrak{A}}d\rangle) = ((a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c)\langle\vee^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c\langle\vee^{\mathfrak{A}}d\rangle)$, and so, as $\mathfrak{A} \models (4.2(\|4.6))[x_0/b]$, we get both $(b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})\neg^{\mathfrak{A}}b\langle\vee^{\mathfrak{A}}d\rangle) = (\neg^{\mathfrak{A}}b\langle\vee^{\mathfrak{A}}d\rangle)$, when taking $c = \neg^{\mathfrak{A}}b$, and $(\neg^{\mathfrak{A}}b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b\langle\vee^{\mathfrak{A}}d\rangle)$, when taking $c = b$. Then, as, by Lemma 5.10, $\mathcal{K}^{(\mathrm{W})}_{\langle\mathrm{M}\rangle}$, being true in $\mathfrak{A}$, is so under $[x_0/(a\|(\neg^{\mathfrak{A}})b), x_1/(b\|a)\langle, x_2/d\rangle]$ ⟨and $\mathfrak{A} \models (4.2)[x_0/d]$⟩, we have both $((\neg^{\mathfrak{A}})b\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} (a\langle\vee^{\mathfrak{A}}d\rangle)$ ⟨in which case, when taking $b = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d$ (resp., $b = \neg^{\mathfrak{A}}e$), we get $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))\rangle$ and $(a\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d\rangle) \leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}})b\langle\vee^{\mathfrak{A}}d\rangle)$ ⟨in which case, when taking $b = d$ (resp., $b = e$), we get $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))\rangle$, and so eventually get $a = (\neg^{\mathfrak{A}})b$ ⟨resp., $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e)) \leqslant^{\mathfrak{A}} (d(\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}e))$, since the Σ_+-quasi-identity $\{(x_0 \wedge x_1) \lessapprox x_2, x_1 \lessapprox (x_0 \vee x_2)\} \to (x_1 \lessapprox x_2)$ is true in distributive lattices⟩. This, by Corollary 4.7 (and 5.5) ⟨as well as 5.8⟩, ends the proof. □

Before pursuing, note that, for each $i \in 2$,

$$\epsilon^4_i \triangleq (\{\langle 3 \cdot (1-i), \langle 1-i, 1, 1-i, 1-i\rangle\rangle\} \cup \{\langle j, \langle \epsilon^4_{3:0}(j-i), i\rangle\rangle \mid j \in ((3+i) \setminus i)\}),$$

being an isomorphism from $\mathfrak{D}_{4[,01]}$ onto $\mathfrak{MS}_{4:i[,01]}\restriction\Sigma_{+[,01]}$, is the one from $\mathfrak{K}_{4:i[,01]} \triangleq (\epsilon^4_i)^{-1}[\mathfrak{MS}_{4:i[,01]}]$ onto $\mathfrak{MS}_{4:i[,01]}$, in which case $\epsilon^8_i \triangleq (\epsilon^4_i \times \Delta_2)$ is that from $\mathfrak{K}_{4:i[,01]} \times \mathfrak{B}_{2[,01]}$ onto $\mathfrak{MS}_{4:i[,01]} \times \mathfrak{B}_{2[,01]}$, and so $\epsilon^5_i \triangleq (\epsilon^8_i\restriction\mathfrak{R}^{\mathfrak{K}_{4:i[,01]}})$ is so from

$\Re(\mathfrak{K}_{4:i[,01]})$ onto $\Re(\mathfrak{MS}_{4:i[,01]})$, the former [bounded] MS lattices being preferably used below due to their having more transparent representation/notation of elements than those of the latter ones. Likewise, $\epsilon^5_i \triangleq \{k+l \mid \langle k,l\rangle \in \Re^{\mathfrak{K}_{4:i[,01]}}\}$, being clearly injective, is an isomorphism from $\Re(\mathfrak{K}_{4:1[,01]})$ onto $\mathfrak{K}_{5:1[,01]} \triangleq \Re(\mathfrak{K}_{4:1[,01]})$ with $(\mathfrak{K}_{5:1[,01]}\restriction\Sigma_{+[,01]}) = \mathfrak{D}_{5[,01]}$. Finally, let $\mathfrak{K}_{2(:0)[,01]} \triangleq (\pi_0\restriction MS_2)[\mathfrak{MS}_{2[,01]}]$, in which case $\epsilon^4_4 \triangleq (((\pi_0\restriction 2^2)(\pi_0\restriction MS_2)^{-1}) \times (\pi_1\restriction 2^2))$ is an isomorphism from $\mathfrak{K}_{2[,01]} \times \mathfrak{B}_{2[,01]}$ onto $\mathfrak{MS}_{2[,01]} \times \mathfrak{B}_{2[,01]}$, and so $\epsilon^3_3 \triangleq (\epsilon^4_4\restriction\Re^{\mathfrak{K}_{2[,01]}})$ is the one from $\Re(\mathfrak{K}_{2[,01]})$ onto $\Re(\mathfrak{MS}_{2[,01]})$, $(\epsilon^4_{3:1})^{-1}$ being that from $\Re(\mathfrak{K}_{2[,01]})$ onto $\mathfrak{K}_{3:0[,01]} \triangleq (\epsilon^4_{3:1})^{-1}[\Re(\mathfrak{K}_{2[,01]})]$.

Theorem 5.12. *Let* $\mathsf{V} \subseteq \mathsf{[B/](W)K\{M\}SL[/A]}$ *be a variety and* $\mathsf{K} \triangleq (\mathsf{MS}_{\mathsf{V}[,01]} \cap \{\mathfrak{B}_{2[,01]}, \mathfrak{S}_{3[,01]}\{,\mathfrak{DM}_{4[,01]}, \mathfrak{K}_{3[,01]}\}(,\mathfrak{K}_{2[,01]})\})$. *Then,* $\mathsf{Q} \triangleq \mathsf{(W)\{M\}RV}$ *is the pre-//quasi-variety generated by* $\Re_{(\mathrm{W})}[\mathsf{MS}_{\mathsf{V}[,01]}\backslash\mathsf{K}] \cup \mathsf{K}$, $\lfloor\mathsf{(W)}\wr\rfloor\{\lceil\mathsf{M}\rceil|\}((\mathsf{R[B/]}\lfloor\mathsf{A}\wr\rfloor(\{\langle\mathsf{Q}\wr\mathsf{P}\rangle\mathsf{S}\}(\mathsf{K}\{\lceil\|\mathsf{M}\rceil\})\{\mathsf{S}\})\lfloor|\mathsf{S}|\mathsf{B}\rfloor)(\mathsf{L[/A]})$ *being the one generated by* $\{\mathfrak{K}_{4\{\langle+1:1\wr0\rangle\}[,01]}|\mathfrak{S}_{3[,01]}|\mathfrak{B}_{2[,01]}\}\{\cup(\{\mathfrak{S}_{3[,01]}\}\langle\cap\varnothing\rangle)\langle\cup(\varnothing\wr(\{\mathfrak{S}_{3[,01]}\}[\cup(\{\mathfrak{K}_{3:0,01}\}/\varnothing)]))\rangle\lceil\cup(\{(\mathfrak{K}\|\mathfrak{M})_{(3\|4)[,01]}\}|\varnothing|\varnothing)\rceil\}\lfloor\cup(\{\mathfrak{K}_{3(-1):0[,01]}\}[/\varnothing])\rfloor$.

Proof. Consider any finitely-generated

$$\mathfrak{A} \in (\mathsf{Q} \setminus (\mathsf{[B]OMSL}(\cup\mathsf{[B]TNIMSL})\{\lceil\cup\mathsf{[B](A)DML}\rceil\}).$$

Take any $\bar a \in A^+$ such that $\mathfrak{A}$ is generated by $\operatorname{img}\bar a$. Let $n \triangleq (\operatorname{dom}\bar a) \in (\omega\setminus 1)$ and $b \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\boldsymbol{\iota}^{\mathfrak{A}}_n(\bar a)$, in which case, by (4.6) and the left alternative of Claim 5.2, we have $\neg^{\mathfrak{A}}b \leqslant^{\mathfrak{A}} b$. Consider any $\mathfrak{B} \in \mathsf{K}' \triangleq (\{\mathfrak{MS}_{6[,01]}\}[\cup(\{\mathfrak{MS}_{2,01}\}/\varnothing)])$ and $h \in \hom(\mathfrak{A},\mathfrak{B})$ $\lceil$such that $(\operatorname{img} h) \not\subseteq (\operatorname{img}\epsilon^6_4)$, in which case, for some $i \in n$, $h(a_i) \notin (\operatorname{img}\epsilon^6_4)$, and so $\pi_0(h(a_i)) = 0 = (1-\pi_0(h(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a_i)))\rceil$. Let $(I|J) \triangleq \{j \in n \mid h(a_j) \notin (\mathcal{F}|\mathcal{I})^{\mathfrak{B}}_{(\mathrm{W})}\}$, $(\imath|\jmath) = |(I|J)|$ and $\bar{\Bbbk}|\bar\ell$ any bijection from $\imath|\jmath$ onto $I|J$. We prove, by contradiction, that there is some $g \in \hom(\mathfrak{A},\mathfrak{B}_{2[,01]})$ such that $g[\operatorname{img}((\bar{\Bbbk}|\bar\ell)\circ\bar a)] = \{0|1\}$. For suppose that, for every $g \in \hom(\mathfrak{A},\mathfrak{B}_{2[,01]})$, there is either some $i' \in \imath$ or some $j' \in \jmath$ such that $g(a_{(\Bbbk|\ell)_{i'|j'}}) = (1|0)$, in which case, as, by Lemma 5.1 and Corollary 5.11, $\hom(\mathfrak{A},\mathfrak{B}_{2[,01]}) \neq \varnothing$, we have $(I\cup J) \neq \varnothing$, and so we are allowed to put $c \triangleq (\vee^{\mathfrak{A}}_+((\bar{\Bbbk}\circ\bar a(\circ\neg^{\mathfrak{A}}\circ\neg^{\mathfrak{A}})) * (\bar\ell\circ\bar a\circ\neg^{\mathfrak{A}})))$. Then, $\pi_{0\wr2}(h((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c)) = 0$, in which case (by (4.6)) $\pi_0(h(\neg^{\mathfrak{A}}c)) = 1$, and so $(\neg^{\mathfrak{A}}c\lceil\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a_i\rceil) \not\leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c\lceil\vee^{\mathfrak{A}}a_i\rceil)$, for $(h\circ\pi_0) \in \hom(\mathfrak{A}\restriction\Sigma_+,\mathfrak{D}_2)$. Now, consider any $\mathfrak{C} \in \mathsf{K}'$, $f \in \hom(\mathfrak{A},\mathfrak{C})$ and the following complementary cases:

- $(\operatorname{img} f) \subseteq (\operatorname{img}\epsilon^5_3)$,
  in which case, by (4.21), $e \triangleq (f\circ(\epsilon^5_3)^{-1}\circ\chi^{3\backslash2}_3) \in \hom(\mathfrak{A},\mathfrak{B}_{2[,01]})$, while $(\operatorname{img} f) \not\subseteq MS_2$, for $(MS_2 \cap (\operatorname{img}\epsilon^5_3)) = \varnothing \neq (\operatorname{img} f)$, as $A \neq \varnothing$, and so $[\mathfrak{C} = \mathfrak{MS}_{6,01}$, whereas], by the assumption to be disproved, $\pi_{1\wr2}(f(c)) = e(c) = 1$. Then, $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c) = \langle 0,0,0\rangle \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$.
- $(\operatorname{img} f) \not\subseteq (\operatorname{img}\epsilon^5_3)$,
  in which case, by (4.6) and the right alternative of Claim 5.2, $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c) \leqslant^{\mathfrak{C}} f(b) = f(\neg^{\mathfrak{A}}b) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$.

Thus, anyway, $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$, in which case, by (2.8) and Theorem 4.4 [resp., Corollary 4.5], $(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$, and so $\mathfrak{A} \not\models \mathcal{R}^{(\mathrm{W})}_{[\mathrm{M}]}[x_0/b, x_1/c\lceil, x_2/a_i\rceil]$. This contradiction to the (weak) $\lceil$Morgan-$\rceil$regularity of $\mathfrak{A}$ definitely shows that, for each $\mathfrak{D} \in ((\mathsf{MS}_{\mathsf{V}[,01]} \setminus \mathsf{K}) \subseteq \mathbf{IS}\mathsf{K}'$ and every $h' \in \hom(\mathfrak{A},\mathfrak{D})$, there is some $g' \in \hom(\mathfrak{A},\mathfrak{B}_2)$ such that $(\operatorname{img} f') \subseteq \Re^{\mathfrak{D}}_{(\mathrm{W})}$, where $f' \triangleq (h'\odot g')$, in which case, by (2.7), $f' \in \hom(\mathfrak{A},\Re_{(\mathrm{W})}(\mathfrak{D}))$, while, by (2.1), $(\ker f') \subseteq (\ker h')$, and so the locality of quasi-varieties, (2.8), (4.2), (4.13), Corollaries 4.7 and 5.11 $\lceil$as well as the injectivity of $\epsilon^6_4\rceil$ complete the argument. □

This, by Lemma 5.1 and Corollary 5.4, immediately yields:

Corollary 5.13. NIMR[B]QS(M|K)SL *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, (\mathfrak{M}|\mathfrak{K})_{(4|3)[,01]} \times \mathfrak{B}_{2[,01]}\}$.

This, in it turn, by Corollaries 4.7, 5.4, 5.6, 5.8, 5.11 and Theorem 5.12, immediately yields:

Corollary 5.14. NIMR[B]QSMSL ∪ (MR)[B](QS){W}K(S)L *is the sub-quasi-variety of* MR[B]QSMSL *relatively axiomatized by either* $\{(5.6), (\neg x_2 \approx x_2) \to \mathcal{K}\}$ *or either version of* (5.5) *and is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{DM}_{4[,01]} \times \mathfrak{B}_{2[,01]}\}$.

Thus, the apparatus of (weak) regularizations of [bounded] (weakly) Kleene-Stone lattices involved in proving Theorem 5.12 yields a more transparent and immediate insight/proof into/to [21, Proposition 4.7]. And what is more, it is involving $\neg\neg\boldsymbol{\iota}_n$ instead of $\wedge_+\langle\boldsymbol{\mu}(x_i)\rangle_{i\in n}$, like therein, that has proved crucial for proving the $\lceil\rceil$-optional version of Theorem 5.12 {though the former choice would suffice for proving the non-optional one, in its turn, sufficient within the framework of [B]SMSL; cf. the final inclusion in Corollary 5.11}, in its turn, yielding axiomatizations of the quasi-equational joins of RQSKSL and all sub-quasi-varieties of DML not subsumed by RKL ⊆ RQSKSL (cf. [18] for latter ones), and so eventual finding the lattice of quasi-varieties of quasi-strong MS lattices, being equally due to the series of "embedability" lemmas presented in Subsection 5.4 (aside from the above basic one 5.3 and that 5.23 to be presented before, in the next subsection, because of its extra meaning related to "nearly De Morgan" framework) as well as "generation/axiomatization" corollaries|theorems presented above|below.

5.3. **"Virtually" versus "almost".** Almost {"⟨quasi-⟩strong "|}"Morgan/Kleene{-Stone}"|"Stone‖Boolean" lattices, satisfying the {}-non-optional version of (the ()-optional one of) the Σ^-_+-quasi-identity (4.13), are called *(practically) virtually {"⟨quasi-⟩strong "|}"Morgan/Kleene{-Stone}"|"Stone‖Boolean" lattices*, their quasi-variety being denoted by

$$\begin{aligned}(\mathsf{P})\mathsf{V}\{\langle\mathsf{Q}\rangle\mathsf{S}|\}(((((\lceil\mathsf{D}\rceil\mathsf{M})/\mathsf{K})\{\mathsf{S}\})|(\mathsf{S}\|\mathsf{B}))\mathsf{L} \supseteq \\ (\mathsf{V}\{\langle\mathsf{Q}\rangle\mathsf{S}|\}(((((\lceil\mathsf{D}\rceil\mathsf{M})/\mathsf{K})\{\mathsf{S}\})|(\mathsf{S}\|\mathsf{B}))\mathsf{L} \supseteq) \\ \{\langle\mathsf{Q}\rangle\mathsf{S}|\}(((((\lceil\mathsf{D}\rceil\mathsf{M})/\mathsf{K})\{\mathsf{S}\})|(\mathsf{S}\|\mathsf{B}))\mathsf{L}.\end{aligned}$$

Note that neither version of

$$\{\neg\neg x_0 \approx \neg x_0(, (x_0 \vee \neg\neg x_0) \approx (\neg\neg x_0 \vee \neg x_0))\} \to (x_0 \approx \neg x_0). \tag{5.7}$$

is true in $\mathfrak{K}_{2(:0)[,01]}$ under $[x_0/0]$. On the other hand, by (4.2), we immediately have:

$$([\mathsf{B}]\mathsf{MSL} \cap \mathrm{Mod}(((5.7)))) = ([\mathsf{B}]\mathsf{MSL} \cap \mathrm{Mod}((5.7))). \tag{5.8}$$

Likewise, by (4.1), we have:

$$([\mathsf{B}]\mathsf{TNIMSL} \cap \mathrm{Mod}((5.7))) = [\mathsf{B}]\mathsf{OMSL}. \tag{5.9}$$

Lemma 5.15. *Let* K ⊆ [B]MSL *and* S|R *its relative sub-quasi-variety relatively axiomatized by (the ()-optional version of) "the {}-non-optional version of"|* ((4.13)|(5.7)). *Then,* R = (NIK ∪ S).

Proof. First, the fact that (the optional version of) (5.7) is equal to (a logical consequence of) $(5.1)[x_i/\neg^{1-i}x_0]_{i\in 2}$ yields the inclusion NIK ⊆ R. Likewise, consider any $\mathfrak{A} \in \mathsf{S}$ and any $a \in \Im^{\mathfrak{A}}_{\neg}$ (such that $(a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) = (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$), in which case $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a$, and so both $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a$ and, by (4.2), $a \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} a$. Then, $\neg^{\mathfrak{A}} a = a$, in which case $\mathfrak{A} \in \mathsf{R}$, and so S ⊆ R. Conversely, consider any $\mathfrak{B} \in \mathsf{IR}$ and any $b \in B$ such that $\neg^{\mathfrak{B}} b \leqslant^{\mathfrak{B}} \neg^{\mathfrak{B}}\neg^{\mathfrak{B}} b$. Take any $c \in \Im^{\mathfrak{B}} \neq \varnothing$, in which case, by (4.1), (4.5) and (4.6) (as well as (4.2)), $(d\|e) \triangleq ((c \vee^{\mathfrak{B}} \neg^{\mathfrak{B}} b) \wedge^{\mathfrak{B}} (\neg^{0\|2})^{\mathfrak{B}} b) =$

$(d\|\neg^{\mathfrak{B}}(d/e)(=(d\|((\neg^{0/2})^{\mathfrak{B}}d\vee^{\mathfrak{B}}\neg^{\mathfrak{B}}d)))$, and so $\neg^{\mathfrak{B}}b\leqslant^{\mathfrak{B}}e=d\leqslant^{\mathfrak{B}}b$. Thus, $\mathfrak{B}\in\mathsf{S}$, as required. □

Given any $\mathfrak{A}\in([\mathsf{QS}]\mathsf{MSL}[\{\backslash\mathsf{OMSL}\}])$ with {zero a and} any ideal I of $\mathfrak{A}\restriction\Sigma_+$ [disjoint with $\mathcal{F}^{\mathfrak{A}}_{(\mathrm{W})}$], $(A\uplus I)\triangleq((\{1\}\times A)\cup(\{0\}\times I))$ forms a subalgebra of $\mathfrak{K}_2\times\mathfrak{A}$, in which case [by Corollary 4.7] $\mathfrak{A}\preceq(\mathfrak{A}\uplus I)\triangleq((\mathfrak{K}_2\times\mathfrak{A})\restriction(A\uplus I))$ is a [virtually quasi-strong] MS-lattice, because, by (2.1) and (2.7), $(A\times\{1\})\times\Delta_A$ is an embedding of $\mathfrak{A}$ into $\mathfrak{A}\uplus I$, as $1\in\Im^{\mathfrak{K}_2}$ [while (4.13) is true in $\mathfrak{A}$] {and so is $(\mathfrak{A}\oplus 1)\triangleq(\mathfrak{A}\uplus\{a\})\in\mathbf{S}(\mathfrak{A}\uplus I)$, since $I\ni a[\notin\mathcal{F}^{\mathfrak{A}}$, for, otherwise, for all $b\in A$, by (4.2), (4.3) and the inequality $a\leqslant^{\mathfrak{A}}\neg^{\mathfrak{A}}b$, we would have $a\leqslant^{\mathfrak{A}}b\leqslant^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b\leqslant^{\mathfrak{A}}\neg^{\mathfrak{A}}a\leqslant^{\mathfrak{A}}a$ implying $b=a$, contrary to the assumption $|A|\neq 1]\}$, whereas $\mathfrak{K}_2\in\mathbf{H}(\mathfrak{A}\uplus I)$, for $\pi_0[A\uplus I]=2$, as $I\neq\varnothing$. Then, $\mathfrak{A}$ is said to be *normal*, if $(A\setminus\mathcal{F}^{\mathfrak{A}}_{\mathrm{W}})[=(A\setminus\mathcal{F}^{\mathfrak{A}})]$ is an ideal of the lattice $\mathfrak{A}\restriction\Sigma_+$ [{with zero}], in which case $[\{(\mathfrak{A}\oplus 1)\in\mathbf{S}\}](\mathfrak{A}\ominus\lrcorner)\triangleq(\mathfrak{A}\uplus(A\setminus\mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}))\in[\mathsf{VQS}]\mathsf{MSL}$, the class of all normal members of any $\mathsf{K}\subseteq\mathsf{MSL}$ being denoted by $\aleph\mathsf{K}$, and so this is a relatively abstract subclass of K including $\mathsf{K}\cap((\mathsf{MS}\setminus\{\mathfrak{MS}_2\})\cup\mathfrak{R}_{\mathrm{W}}[\mathsf{WKSL}])$, because, for any $\mathfrak{B}\in(\mathsf{MS}\setminus\{\mathfrak{MS}_2\})$, $B\setminus\mathcal{F}^{\mathfrak{B}}_{\mathrm{W}}$ is the singleton constituted by the zero of the finite lattice $\mathfrak{B}\restriction\Sigma_+$ that implies $(\mathfrak{B}\ominus\lrcorner)=(\mathfrak{B}\oplus 1)$, while, for any $\mathfrak{C}\in\mathsf{WKSL}$, $(\mathfrak{R}^{\mathfrak{C}}_{\mathrm{W}}\setminus\mathcal{F}^{\mathfrak{R}_{\mathrm{W}}(\mathfrak{C})}_{\mathrm{W}})=(\mathcal{I}^{\mathfrak{C}}_{\mathrm{W}}\times 1)$ that, by (2.1) and (2.7), implies:

(5.10) $((\pi_1\otimes(\pi_0\times(\pi_1\circ\pi_1)))\restriction(\mathfrak{R}_{\mathrm{W}}(C)\ominus\lrcorner))\in\hom_{\mathrm{I}}(\mathfrak{R}_{\mathrm{W}}(\mathfrak{C})\ominus\lrcorner,\mathfrak{R}_{\mathrm{W}}(\mathfrak{C})\times(\mathfrak{B}_2\oplus 1))$.

And what is more, by Corollary 4.7, (5.9) and Lemma 5.15, we immediately have:

Corollary 5.16. $(\mathfrak{B}_2\oplus 1)\in(\mathsf{VBL}\setminus\mathsf{BL})$*, in which case* VBL*,* BL*,* ABL*,* TNIMSL *and* OMSL *form a pentagon of the lattice of quasi-varieties of almost Boolean lattices, and so this is not distributive.*

Likewise, by Corollary 4.7, $\mathfrak{S}_3\preceq\mathfrak{MS}_{5:1}\triangleq((\mathfrak{K}_2\times\mathfrak{S}_3)\restriction((2\times 3)\setminus\{\langle 0,2\rangle\}))\in(\mathsf{PVSL}\setminus\mathsf{VSL})$, for $\mathfrak{K}_2\in\mathsf{ASL}\ni\mathfrak{S}_3\preceq(\mathfrak{S}_3\oplus 1)\in\mathbf{S}\mathfrak{MS}_{5:1}$, while (4.13) is true in $\mathfrak{S}_3\oplus 1$, whereas $\mathfrak{MS}_{5:1}\not\models((4.9))[x_0/\langle 0,1\rangle]$ as well as $(MS_{5:1}\setminus(S_3\oplus 1))=\{\langle 0,1\rangle\}$, but $\mathfrak{MS}_{5:1}\not\models(4.13)[x_0/\langle 0,1\rangle]$.

Theorem 5.17. *Let* $\mathsf{K}\subseteq(\aleph\mathsf{MSL}(\cap\mathbf{IS}((\mathsf{MS}\cap\mathsf{QSMSL})\otimes\{\mathfrak{B}_2^j\mid j\in 2\})))$*,* $(\mathsf{P})\mathsf{VP}$ *the relative sub-quasi-variety of* $\mathsf{P}\triangleq\mathbf{PV}(\mathsf{K}\cup\{\mathfrak{K}_2\})$*, relatively axiomatized by the* $\{\}$*-non-optional version of (the ()-optional one of) the* Σ^-_+*-quasi-identity* (4.13)*, and* $\mathsf{K}'\triangleq(((\mathsf{K}(\backslash\{\mathfrak{S}_3\}))\ominus\lrcorner)(\cup\{\mathfrak{MS}_{5:1}\mid\mathfrak{S}_3\preceq\mathsf{K}\}))$*. Suppose both* K *and all its members are finite [while* $\mathsf{P}\not\subseteq\mathsf{TNIMSL}$*]. ⌊In particular,* $\mathsf{K}=\mathsf{MS}_{\mathsf{V}}$*, where* $\mathsf{V}\triangleq\{\langle\mathsf{Q}\rangle\mathsf{S}|\}((\mathsf{M}/\mathsf{K})\{\mathsf{S}\})|(\mathsf{S}\|\mathsf{B}))\mathsf{L}$*, in which case* $\mathsf{K}'=(((\mathsf{K}(\backslash\{\mathfrak{S}_3\}))\oplus 1)(\cup\{\mathfrak{MS}_{5:1}\mid\mathfrak{S}_3\preceq\mathsf{K}\}))$*, while* $\mathsf{P}=\mathsf{AV}$*, and so* $(\mathsf{P})\mathsf{VP}=(\mathsf{P})\mathsf{VV}⌋$*. Then,* $(\mathsf{P})\mathsf{VP}[\cup\mathsf{NIP}]$ *is the pre-//quasi-variety generated by* $\mathsf{K}'[\cup\{\mathfrak{K}_2\times\mathfrak{B}_2\}]$*.*

Proof. Consider any non-one-element finitely-generated $\mathfrak{A}\in(\mathsf{P})\mathsf{VP}$ and any $g\in\hom(\mathfrak{A},\mathfrak{K}_2)$, in which case $\mathfrak{A}$ is finite, for $\mathsf{MSL}\ni\mathfrak{A}$, being finitely-generated, in view of Theorem 4.4, is locally-finite, and so is $\mathrm{Co}(\mathfrak{A})\supseteq\mathcal{C}\triangleq\ker[\hom(\mathfrak{A},\mathsf{K})]$, while, by (2.8), (5.9), Corollary 4.7 and Lemma 5.15, $\mathcal{C}\neq\varnothing$, for:

$$\Delta_A=(A^2\cap(\bigcap(\ker[\hom(\mathfrak{A},\mathfrak{K}_2)]\cup\mathcal{C})),\tag{5.11}$$

and so there are some bijections $\bar{a}$ from $\Bbbk\triangleq|A|\in(\omega\setminus 1)$ onto A and $\bar{\theta}$ from $n\triangleq|\mathcal{C}|\in(\omega\backslash 1)$ onto $\mathcal{C}$ as well as both some $\overline{\mathfrak{B}}\in\mathsf{K}^n$ and some $\bar{h}\in\prod_{i\in n}\hom(\mathfrak{A},\mathfrak{B}_i)$ such that $\bar{\theta}=(\bar{h}\circ\ker)$. Let us prove, by contradiction, that $\mathcal{A}\triangleq(\prod_{j\in n}(g^{-1}[1]\cap h_j^{-1}[\mathcal{F}^{\mathfrak{B}_j}_{\mathrm{W}}(\cap\{c\in B_j\mid(c\vee^{\mathfrak{B}_j}\neg^{\mathfrak{B}_j}c)=(\neg^{\mathfrak{B}_j}\neg^{\mathfrak{B}_j}c\vee^{\mathfrak{B}_j}\neg^{\mathfrak{B}_j}c),h_j[\mathfrak{A}]\in\mathbf{I}\mathfrak{S}_3\})]))=\varnothing$. For suppose $\mathcal{A}\neq\varnothing$, in which case, by its finiteness, ensuing from that of A and the inclusion $\mathcal{A}\subseteq A^n$, there is a bijection $\bar{\boldsymbol{a}}$ from $m\triangleq|\mathcal{A}|\in(\omega\setminus 1)$ onto $\mathcal{A}$, and so we have $\bar{b}\triangleq\langle\vee^{\mathfrak{A}}_{+}\boldsymbol{a}_k\rangle_{k\in m}\in A^m$ such that $(\bar{b}\circ g)=(m\times 1)$. Then, by induction on any

$l \in ((m+1) \setminus 1) \ni m$, we get $g(\boldsymbol{\iota}^{\mathfrak{A}}_{x_0,l}(\bar{b}\restriction l)) = 0$, in which case $d \triangleq \boldsymbol{\iota}^{\mathfrak{A}}_{x_0,m}(\bar{b}) \in g^{-1}[1]$, and so $\neg^{\mathfrak{A}} d \not\leqslant^{\mathfrak{A}} d$, for $g(d) = 0 \not\geqslant 1 = g(\neg^{\mathfrak{A}} d)$. On the other hand, for all $\imath \in n$ and $\jmath \in m$, $\pi_\imath(\boldsymbol{a}_\jmath) \leqslant^{\mathfrak{A}} b_\jmath$, in which case $\mathcal{F}^{\mathfrak{B}_\imath}_{\mathrm{W}} \ni h_\imath(\pi_\imath(\boldsymbol{a}_\jmath)) \leqslant^{\mathfrak{B}_\imath} h_\imath(b_\jmath)$, and so, applying (4.3) twice, we get $h_\imath(b_\jmath) \in \mathcal{F}^{\mathfrak{B}_\imath}_{\mathrm{W}}$. Hence, by the truth of (4.14) in $\mathfrak{K}_2$ and (5.11), $b_\jmath \in \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}$, in which case, by Claim 5.2, $d \in \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}(\ni d' \triangleq \boldsymbol{\iota}_{\Bbbk}(\bar{a}))$, and so $\mathfrak{A} \not\models (4.13)[x_0/d]$ (as well as, by (5.3), $g' \triangleq \boldsymbol{\pi}^{\mathfrak{A}}(d', d) \in \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}$. Consider any $\imath \in n$ and the following complementary cases, putting $\mathfrak{C} \triangleq h_\imath[\mathfrak{A}]$:

- $\mathfrak{C} \preceq \mathfrak{S}_3$,
  in which case there is an injective $e \in \hom(h_\imath[\mathfrak{A}], \mathfrak{S}_3)$, while $f \triangleq (h_\imath \circ e) \in \hom(\mathfrak{A}, \mathfrak{S}_3)$, whereas, by induction on any $\ell \in ((\Bbbk+1) \setminus 1) \ni \Bbbk$, we have $f(\boldsymbol{\iota}^{\mathfrak{A}}_{\ell}(\bar{a}\restriction \ell) = 2$, and so, in particular, when $\ell = \Bbbk$, we get $f(d') = 2$. Consider the following complementary subcases:
  - $(\operatorname{img} e) = 3$,
    in which case, for each $\jmath \in m$, by the injectivity of e, we have $(3 \setminus 1) \ni f(\pi_\imath(\boldsymbol{a}_\jmath)) \neq 1$, and so, since $\pi_\imath(\boldsymbol{a}_\jmath) \leqslant^{\mathfrak{A}} b_\jmath$, we get $2 = f(\pi_\imath(\boldsymbol{a}_\jmath)) \leqslant f(b_\jmath) \leqslant 2$. Then, by induction on any $l \in ((m+1) \setminus 1) \ni m$, we get $f(\boldsymbol{\iota}^{\mathfrak{A}}_{x_0,l}(\bar{b}\restriction l) = 2$, for $(\bar{b} \circ f) = (m \times \{2\})$, in which case $f(d) = 2$, and so $f(g') = 2$. Therefore, $f(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = f(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$.
  - $(\operatorname{img} e) \neq 3$,
    in which case $1 \notin (\operatorname{img} e) = (\operatorname{img} f) \ni f(g')$, for $(\neg^{1\wr 2})^{\mathfrak{S}_3} 1 = (0 \wr 2)$, and so $f(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = f(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$.

  Thus, in any case, $f(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = f(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$, and so, by the injectivity of e, $h_\imath(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = h_\imath(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$.
- $\mathfrak{C} \not\preceq \mathfrak{S}_3$,
  in which case there are some $\mathfrak{D} \in \mathbf{S}\mathfrak{M}\mathfrak{S}_6$ with $a' \triangleq \langle 0,1,0\rangle \notin D$ and some $j \in 2$ such that $\mathfrak{C} \preceq (\mathfrak{D} \times \mathfrak{B}^j_2)$, and so $\mathfrak{E} \triangleq (\pi_0\restriction C)[\mathfrak{C}] = f'[\mathfrak{A}] \not\preceq \mathfrak{S}_3$, where $f' \triangleq (h_\imath \circ \pi_0) \in \hom(\mathfrak{A}, \mathfrak{M}\mathfrak{S}_6)$, for, otherwise, as $(\mathfrak{D} \times \mathfrak{B}^0_2) \in \mathbf{I}\mathfrak{D}$, while $\mathfrak{B}_2 \preceq \mathfrak{S}_3$, whereas $(\pi_{0\wr 1}\restriction C) \in \hom(\mathfrak{C}, \mathfrak{D} \wr \mathfrak{B}^j_2)$, by (2.1) and (2.7), we would have $\mathfrak{C} \in \mathbf{S}(\mathfrak{E} \times \mathfrak{B}^j_2) \preceq \mathfrak{S}_3$. Then, $(\operatorname{img} f') = E \not\subseteq (\operatorname{img} \epsilon^5_3)$, for, otherwise, $(\epsilon^5_3)^{-1}\restriction E$ would be an embedding of $\mathfrak{E}$ into $\mathfrak{S}_3$, in which case, by the right alternative of Claim 5.2, $f'(d') \in \Im^{\mathfrak{M}\mathfrak{S}_6}_{\neg}$, and so, since $f'(d) \in \mathcal{F}^{\mathfrak{M}\mathfrak{S}_6}_{\mathrm{W}}$, for $d \in \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}$, by (5.3), $f'(g') \in (\Im^{\mathfrak{M}\mathfrak{S}_6}_{\neg} \setminus \{a'\}) = \Im^{\mathfrak{M}\mathfrak{S}_6}$, for $a' \notin D \supseteq E = (\operatorname{img} f') \ni f'(g')$. In that case, $f'(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = f'(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$, and so, by the truth of ((4.9)) in $\mathfrak{B}^j_2$, $h_\imath(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = h_\imath(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$.

Thus, anyway, $h_\imath(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = h_\imath(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$, in which case, by the truth of ((4.9)) in $\mathfrak{K}_2$ and (5.11), $(g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g') = (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} g' \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} g')$, and so $\mathfrak{A} \not\models ((4.13))[x_0/g']$, for $\neg^{\mathfrak{A}} g' \not\leqslant^{\mathfrak{A}} g'$, because $g(g') = 0 \not\geqslant 1 = g(\neg^{\mathfrak{A}} g')$, as $g(d) = 0$.) This contradiction shows that there is some $i \in n$ such that (providing there is some $e'' \in \hom^{\mathrm{S}}_{\mathrm{I}}(h_i[\mathfrak{A}], \mathfrak{S}_3))$, for all $b' \in A$, either $g(b') = 0$ or $h_i(b') \notin \mathcal{F}^{\mathfrak{B}_i}_{\mathcal{F}}$ (i.e., $e''(h_i(b') = 0$, or $(h_i(b') \vee^{\mathfrak{B}_i} \neg^{\mathfrak{B}_i} h_i(b') \neq (\neg^{\mathfrak{B}_i} h_i(b') \vee^{\mathfrak{B}_i} \neg^{\mathfrak{B}_i} h_i(b'))$, i. e. $e''(h_i(b') = 1)$, while, by (2.7), $h' \triangleq (g \odot (h_i(\circ e'')) \in \hom(\mathfrak{A}, \{\mathfrak{K}_2\} \otimes (\{\mathfrak{C}_i\}(\cap\varnothing))(\cup\{\mathfrak{S}_3\}))$, in which case $(\operatorname{img} h') \subseteq (((C_i \ominus \beth)(\cap\varnothing))(\cup MS_{5:1}))$ (whereas $\mathfrak{S}_3 \preceq \mathsf{K}$), and so $h' \in \hom(\mathfrak{A}, \mathsf{K}'(\cup(\mathfrak{S}_3 \oplus 1)))$, as well as, by (2.1), $(\ker h') \subseteq (\ker g)$. In this way, (2.8), Corollary 4.7, the inclusion(s) $\mathsf{K} \subseteq \mathbf{IS}(\mathsf{K}')$ (and $\{\mathfrak{S}_3 \oplus 1\} \subseteq \mathbf{S}\{\mathfrak{M}\mathfrak{S}_{5:1}\}$), the locality of quasi-varieties and the quasi-equationality of finitely-generated pre-varieties [as well as Lemmas 5.3, 5.7, 5.15 and Corollary 5.4] complete the argument. □

5.3.1. *Enhanced generation.* Recall the following immediate observation:

Lemma 5.18. *Let $\mathfrak{A} \in \mathsf{A}_\wedge$ be a semi-lattice with bound $b \in A$ (i.e., $b = (a \wedge^{\mathfrak{A}} b)$, for all $a \in A$) and $h : A \to (\operatorname{img} h)$. Suppose $(\ker h) \in \mathrm{Co}(\mathfrak{A})$. Then, $h[\mathfrak{A}]$ is a semi-lattice with bound $h(b)$.*

Given any $(\mathfrak{A}|\mathfrak{B}) \in \mathsf{MSL}$ with zero $a|b$ of $(\mathfrak{A}|\mathfrak{B})\restriction\Sigma_+$ and any $h \in \hom^{[\mathrm{S}]}_{(\mathrm{I})}(\mathfrak{A}, \mathfrak{B})$ such that [resp., in which case, by Lemma 5.18] $h(a) = b$, [and so] we have:

$$(h \oplus 1) \triangleq ((\pi_0\restriction(A \oplus 1)) \odot ((\pi_1 \circ h)\restriction(A \oplus 1))) \in \hom^{[\mathrm{S}]}_{(\mathrm{I})}(\mathfrak{A} \oplus 1, \mathfrak{B} \oplus 1). \tag{5.12}$$

On the other hand,

$$\hbar_1 \triangleq \{\langle i, \min(2, i)\rangle \mid i \in 4\} \in \hom^{\mathrm{S}}(\mathfrak{K}_{(4:1[,01]}, \mathfrak{K}_{3[,01]}), \tag{5.13}$$

in which case, by (2.1), (2.7) and (5.12):

$$((\hbar_1 \oplus 1) \odot (\pi_1\restriction(4 \oplus 1))) \in \hom_{\mathrm{I}}(\mathfrak{K}_{4:1} \oplus 1, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{K}_{4:1}), \tag{5.14}$$

and so, by Lemma 4.6, Corollary 4.7, (5.12) and Theorem 5.17, we immediately get the following quite useful enhancement of the latter:

Corollary 5.19. *$([\mathsf{NIA}\{\mathsf{QS}\}(\mathsf{M}|\mathsf{K})\{\mathsf{S}\}\mathsf{L}\cup]\langle\mathsf{P}\rangle\mathsf{V}\{\mathsf{QS}\}(\mathsf{M}|\mathsf{K})\{\mathsf{S}\}\mathsf{L})$ is the pre-/quasi-variety generated by $\{(\mathfrak{M}|\mathfrak{K})_{4|3} \oplus 1\{, \mathfrak{K}_{4:1}\langle, \mathfrak{M}\mathfrak{S}_{5:1}\rangle\}[, \mathfrak{K}_2 \times \mathfrak{B}_2]\}$.*

Likewise, by (2.1), (2.7), (4.21) and (5.12), we have:

$$((\chi_3^{3\backslash 1} \oplus 1) \odot (\pi_1\restriction(3 \oplus 1))) \in \hom_{\mathrm{I}}(\mathfrak{S}_3 \oplus 1, (\mathfrak{B}_2 \oplus 1) \times \mathfrak{S}_3), \tag{5.15}$$

and so, by Lemma 4.6, Corollary 4.7, (5.12) and Theorem 5.17, we also get one more equally useful enhancement of the latter:

Corollary 5.20. *$([\mathsf{NIA}(\mathsf{S}|\mathsf{B})\mathsf{L}\cup]\langle\mathsf{P}\rangle\mathsf{V}(\mathsf{S}|\mathsf{B})\mathsf{L}) = (\mathbf{P}/\mathbf{Q})\mathbf{V}((\{\langle\mathfrak{M}\rangle\mathfrak{S}_{3\langle+2:1\rangle}\}|\varnothing)\cup\{(\mathfrak{B}_2 \oplus 1)^{1\langle-(1|0)\rangle}[, \mathfrak{K}_2 \times \mathfrak{B}_2]\})$.*

It is Corollary 5.19|5.20 that releases us from proving "embedability lemma" for $((\mathfrak{K}|\mathfrak{S})_{(4:1)|3} \oplus 1)[\times\mathfrak{B}_2|]$ in finding the lattice of quasi-varieties of almost quasi-strong Kleene-Stone lattices.

5.3.2. *Regularity versus virtuality.* First, by (4.2), we clearly have:

$$(\mathsf{WRMSL} \cap \mathrm{Mod}((4.2))) = (\mathsf{RMSL} \cap \mathrm{Mod}((4.2))). \tag{5.16}$$

However, this can not be extended to neither ((4.2)) nor *Morgan* regularity, even within the framework of AQSKSL, because of the instances of $(\mathfrak{M}\mathfrak{S}_{5:1}|(\mathfrak{K}_3 \oplus 1)) \in ((\mathbf{S}(\mathfrak{K}_2 \times (\mathfrak{S}|\mathfrak{K})_3) \setminus | \cap \mathrm{Mod}((4.2))) \cap \mathrm{Mod}(((4.2))))$, being weakly Morgan regular almost quasi-strong Kleene-Stone lattices, for both $\mathfrak{K}_2$ and $(\mathfrak{S}|\mathfrak{K})_3$ are so, but not satisfying $\mathcal{R}_{[\mathrm{M}]}$ under $[x_i/\langle 1 - \min(i, 1), 1 - (0|\min(i, 1))\rangle]_{i\in(2[+1])}$. Nevertheless,

$$\Re(\mathfrak{K}_2) = (\mathfrak{B}_2 \oplus 1), \tag{5.17}$$

in which case, by Theorem 5.12, we get:

$$[\mathsf{M}]\mathsf{RAQSK}[\mathsf{M}]\mathsf{SL} \subseteq (\mathsf{P})\mathsf{VQSK}[\mathsf{M}]\mathsf{SL}, \tag{5.18}$$

while, by Lemmas 5.11, 5.15 and (5.9) (as well as (5.8)), we have:

$$\mathsf{WR}(\mathsf{P})\mathsf{VQSMSL} \subseteq \mathsf{NIMSL}, \tag{5.19}$$

whereas, by (5.12), we also have:

$$(\epsilon_2^3 \oplus 1) \in \hom_{\mathrm{I}}(\mathfrak{B}_2 \oplus 1, \mathfrak{S}|\mathfrak{K}_3 \oplus 1) \subseteq (\hom_{\mathrm{I}}(\mathfrak{B}_2 \oplus 1, \mathfrak{M}\mathfrak{S}_{5:1}|(\mathfrak{K}_3 \oplus 1)), \tag{5.20}$$

in which case, by (2.1) and (2.7), we get:

$$((\epsilon_2^3 \oplus 1) \odot (\pi_1\restriction(B_2 \oplus 1)) \in \hom_{\mathrm{I}}(\mathfrak{B}_2 \oplus 1, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2), \tag{5.21}$$

and so, as $\{\langle i, i + (2 \cdot \min(i, 1))\rangle \mid i \in 3\}$ is an embedding of $\mathfrak{S}_3$ into $\mathsf{K}_{5:1}$, by (5.10), (5.15), Theorems 5.12, 5.17, Corollaries 4.7, 5.4 and Lemma 5.3, eventually get:

Corollary 5.21. $\mathsf{W}[\langle\mathsf{M}|\rangle]\mathsf{R}(\mathsf{P})\mathsf{V}((([\{\mathsf{Q}\}\mathsf{S}](\mathsf{K}[\langle\|\mathsf{M}\rangle])[\mathsf{S}])|(\mathsf{S}/\mathsf{B}))\mathsf{L} = (\mathbf{P}\wr\mathbf{Q})\mathbf{V}(\{\mathfrak{B}_2 \oplus 1\}\cup((\varnothing[\cup\{(\mathfrak{M})\mathfrak{S}_{3(+2:1)}\}])|(\{(\mathfrak{M})\mathfrak{S}_{3(+2:1)}\}/\varnothing))\cup(\{\mathfrak{K}_{4[\{+1:1\}]}^{[\langle 0\{+1\}\rangle]}[\langle, (\mathfrak{K}\|\mathfrak{M})_{3\|4}\oplus 1\rangle]\}|\varnothing))$. *In particular,* $\{\mathsf{NI}\}\mathsf{WR}[\mathsf{M}](\mathsf{P})\mathsf{V}\langle\mathsf{Q}\rangle\mathsf{SKSL} = (\mathbf{P}|\mathbf{Q})\mathbf{V}((\{(\mathfrak{M})\mathfrak{S}_{3(+2:1)}\}\langle\cap\{\mathfrak{M}\mathfrak{S}_{5:1}\}\rangle)\langle\cup\{\mathfrak{K}_{5:1}\}\rangle\cup(((\{\mathfrak{K}_{(4[-1])}\}\langle\cap\{\mathfrak{K}_3\}\rangle)[\oplus 1])[\{\otimes\{\mathfrak{B}_2\}\}])\cup(\{\mathfrak{B}_2\oplus 1\}[\cap\varnothing](\cap\varnothing)))$.

5.3.2.1. Quasi-strong regularity. MS lattices satisfying the Σ_+^--quasi-identity $\mathcal{R}_{[\mathrm{N}]}^{\mathrm{QS}} \triangleq ((\pi_0(\mathcal{R})[\cup\{((4.9))[x_0/(x_0\wedge x_1)]\}]) \to ((\neg x_1 \wedge \neg\neg x_1) \lessapprox x_1))$ are called *[nearly] quasi-strongly regular*, the class of such members of any $\mathsf{K}\subseteq\mathsf{MSL}$ being denoted by $[\mathsf{N}]\mathsf{QSRK}[\supseteq\mathsf{QSRK}]$, in which case:

$$\mathsf{NQSRNDML} = \mathsf{QSRNDML}, \tag{5.22}$$

and so this term is justified by the following analogue of Corollary 5.11, immediately ensuing from the fact that, due to the idempotence identity for $\vee$, we have:

$$(\mathsf{MSL}\cap\mathrm{Mod}(\mathcal{R}_{\mathrm{M}}[x_2/x_1])) = (\mathsf{MSL}\cap\mathrm{Mod}(\mathcal{R}^{\mathrm{QS}})) \tag{5.23}$$

[and the truth of the Σ_+^--quasi-identity:

$$\{\neg x_1 \approx x_1, x_0 \lessapprox x_1\} \to ((4.9)), \tag{5.24}$$

in MS lattices, in its turn, being due to that of (4.2) and (4.3)]:

Lemma 5.22. $(\mathsf{MRMSL}\cup\mathsf{QSMSL})\subseteq[\mathsf{N}]\mathsf{QSRMSL}\subseteq(\mathsf{NIMSL}\cup\mathsf{QSMSL})$.

Likewise, by (4.1), (4.2), (4.5), (4.6) and (4.18), we immediately have:

Lemma 5.23. *Let* $\mathfrak{A}\in\mathsf{MSL}$, $a\in A$, $c \triangleq (a\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}a)$ *and* $d\triangleq(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}a)$. *(Suppose* $c\neq d$.*) Then,* $(c\neq)b\triangleq\neg^{\mathfrak{A}}c=\neg^{\mathfrak{A}}d\leqslant^{\mathfrak{A}} c\leqslant^{\mathfrak{A}} d=\neg^{\mathfrak{A}}b(\neq c)$ *(in which case* $\{\langle 0,b\rangle,\langle 1,c\rangle,\langle 2,d\rangle\}$ *is an embedding of* $\mathfrak{S}_3$ *into* $\mathfrak{A}$*), and so* $\mathfrak{S}_3$ *is embeddable into any member of* $(\mathsf{MSL}\setminus\mathsf{NDML})\supseteq([\mathsf{A}]\mathsf{QSMSL}\setminus[\mathsf{A}]\mathsf{DML})$.

Theorem 5.24. *Let* $\mathsf{K}\subseteq(\aleph)\mathsf{QSMSL}$ *and* $\mathsf{P}=\mathbf{PV}(\mathsf{K}\cup\{\mathfrak{K}_2\})$. *Suppose both* K *and all members of it are finite, while* $\mathfrak{S}_3\preceq|\not\leq\mathsf{K}|$ *"whereas* $\mathsf{K}\not\subseteq\mathsf{OMSL}$*". Then,* $\mathsf{Q}\triangleq[\mathsf{N}]\mathsf{QSRP}$ *is the pre-/quasi-variety generated by* $\mathsf{K}'\triangleq(\mathsf{K}\cup\{((\mathfrak{K}_2\times\mathfrak{S}_3)\restriction((S_3\oplus 1)[\cup MS_{5:1}]))|(\mathfrak{B}_2\oplus 1)\})\subseteq(\aleph)[\mathsf{P}]\mathsf{VQSMSL}$. *In particular,* $\mathsf{WR}[\mathsf{P}]\mathsf{V}\{\mathsf{Q}\}\mathsf{SKSL} = [\mathsf{N}]\mathsf{QSRWR}[\mathsf{P}]\mathsf{V}\{\mathsf{Q}\}\mathsf{SKSL}$.

Proof. Consider any finitely-generated $\mathfrak{A}\in(\mathsf{Q}\setminus\mathsf{QSMSL})$, in which case, by Lemma 5.22 and the local finiteness of MSL, for this is finitely-generated, in view of Theorem 4.4, $\mathfrak{A}$ is both finite, non-one-element and non-idempotent, and so "as $\mathfrak{B}_2\preceq\mathfrak{S}_3$"|, by Lemma 5.1, $\mathcal{H}\triangleq\hom(\mathfrak{A},(\mathfrak{S}|\mathfrak{B})_{3|2})$ is both non-empty and finite. Then, there are some bijections $\bar{a}$ from $n\triangleq|A|\in(\omega\setminus 1)$ onto A and $\bar{h}$ from $m\triangleq|\mathcal{H}|\in(\omega\setminus 1)$. Let $b\triangleq\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\boldsymbol{\iota}_n^{\mathfrak{A}}(\bar{a})$, in which case, by (4.6) and the left alternative of Claim 5.2, $\neg^{\mathfrak{A}}b\leqslant^{\mathfrak{A}} b$. Consider any $g\in\hom(\mathfrak{A},\mathfrak{K}_2)$. By contradiction, we prove that $\mathcal{A}\triangleq(\prod_{i\in n}(g^{-1}[1]\cap h_i^{-1}[(3|2)\setminus((1[+1])|1)]))=\varnothing$. For suppose there is some $\bar{c}\in\mathcal{A}$. Put $d\triangleq(\vee_+^{\mathfrak{A}}\bar{c})$, in which case $g(d)=0$, and so $(\neg^{\mathfrak{A}}d\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d)\not\leqslant d$, for $g(\neg^{\mathfrak{A}}d\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d)=1\not\leqslant 0$. Now, consider any $f\in\hom(\mathfrak{A},\mathfrak{M}\mathfrak{S}_6)$ and the following complementary cases:

- $(\mathrm{img}\, f)\subseteq(\mathrm{img}\,\epsilon_3^5)$,
  in which case |"by (4.21)" $e\triangleq(f\circ(\epsilon_3^5)^{-1}\circ(\Delta_3|\chi_3^{3\setminus 1}))\in\hom(\mathfrak{A},(\mathfrak{S}|\mathfrak{B})_{3|2})$, and so $e=h_j$, for some $j\in n$. Then, as $c_j\leqslant^{\mathfrak{A}} d$, $((1[+1])|1)\leqslant e(c_j)\leqslant e(d)$, in which case $e(\neg^{\mathfrak{A}}d)=0$, and so $f(\neg^{\mathfrak{A}}d\wedge^{\mathfrak{A}} b)=\langle 0,0,0\rangle\leqslant^{\mathfrak{M}\mathfrak{S}_6} f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}} d)$. [And what is more, by induction on any $k\in((n+1)\setminus 1)\ni n$, we have $f(\boldsymbol{\iota}_k^{\mathfrak{A}}(\bar{a}\restriction k))=\langle 1,1,1\rangle$, in which case $f(b)=\langle 1,1,1\rangle=f(d)$, and so $f((b\wedge^{\mathfrak{A}} d)\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}(b\wedge^{\mathfrak{A}} d))=\langle 1,1,1\rangle=f(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(b\wedge^{\mathfrak{A}} d)\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}(b\wedge^{\mathfrak{A}} d))$.]|

- $(\mathrm{img}\, f) \not\subseteq (\mathrm{img}\, \epsilon^5_3)$,
in which case, by (4.6) and the right alternative of Claim 5.2, $f(\neg^{\mathfrak{A}} b) = f(b)$, and so $f(\neg^{\mathfrak{A}} d \wedge^{\mathfrak{A}} b) \leqslant^{\mathfrak{MS}_6} f(b) = f(\neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{MS}_6} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} d)$ [while, by (5.24), $f((b \wedge^{\mathfrak{A}} d) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d)) = f(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d))$.]|

Thus, anyway, by (2.8) and Theorem 4.4 |[as well as Lemma 5.23 and the truth of ((4.9)) in $\mathfrak{K}_2$], we conclude that [both $((b \wedge^{\mathfrak{A}} d) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d)) = (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(b \wedge^{\mathfrak{A}} d))$ and] $(\neg^{\mathfrak{A}} d \wedge^{\mathfrak{A}} b) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} d)$, in which case $\mathfrak{A} \not\models \mathcal{R}^{\mathrm{QS}}_{[\mathrm{N}]}[x_0/b, x_1/d]$, and so this contradiction to the [nearly] quasi-strong regularity of $\mathfrak{A}$ shows that $\mathcal{A} = \varnothing$. In this way, there is some $l \in n$ such that $h_l[g^{-1}[1]] \subseteq ((1[+1])|1)$, in which case, by (2.7), $\hbar \triangleq (g \odot h_l) \in \hom(\mathfrak{A}, \mathsf{K}')$, while, by (2.1), $(\ker \hbar) \subseteq (\ker g)$, and so (2.8), the locality of quasi-varieties, the quasi-equationality of finitely-generated pre-varieties, Corollary 5.21, Lemma 5.22 and "the [nearly] quasi-strong regularity of $(\mathfrak{K}_2 \times \mathfrak{S}_3)\restriction((S_3 \oplus 1)[\cup MS_{5:1}])$ to be checked immediately"|"Lemma 5.3 as well as (5.17)" complete the argument. □

Theorem 5.25. $\mathsf{Q} \triangleq \mathsf{NQSRVQSMSL} = \mathsf{QSRVQSMSL}$. *In particular,* $(\lceil \mathsf{NI} \rceil(\mathsf{N})\mathsf{QSR}[\mathsf{P}]\mathsf{VV}\lfloor \cup(\mathsf{NIU}\lceil \cap \varnothing \rceil)\rfloor) = (\mathbf{P} \wr \mathbf{Q})\mathbf{V}(\{(((\lceil \mathsf{I} \rceil \mathsf{MS}_{\mathsf{V}}\lceil \otimes\{\mathfrak{B}_2\}\rceil)\lceil \cup \mathsf{NIMS}_{\mathsf{V}}\rceil) \cup ((\{\mathfrak{B}_2 \oplus 1\}[(\cap((\mathsf{MSL}\{\cap\varnothing\}|(\varnothing\|\mathsf{MSL})))])[(\cup((\varnothing\{\cup\{\mathfrak{MS}_{5:1}\}\})|(\{\mathfrak{MS}_{5:1}\}\|\varnothing)))])\rfloor\lceil \varnothing \cap \rceil((\mathsf{IMS}_{\mathsf{U}} \otimes \mathfrak{B}_2) \cup \mathsf{NIMS}_{\mathsf{U}})\rfloor)$, *where* $\mathsf{V} \triangleq (((\{\langle \mathsf{Q} \rangle \mathsf{S}\}(\mathsf{M}/\mathsf{K})\{\mathsf{S}\})|(\mathsf{S}\|\mathsf{B}))\lfloor$ *and* $\mathsf{U} \supseteq \mathsf{AV}$ *is a sub-variety of* $\mathsf{MSL}\rfloor$.

Proof. The inclusion from right to left is trivial, for $\mathcal{R}^{\mathrm{QS}}_{\mathrm{N}}$ is a logical consequence of $\mathcal{R}^{\mathrm{QS}}$. Conversely, consider any finitely-generated $\mathfrak{A} \in (\mathsf{Q} \setminus \mathsf{QSMSL}) \subseteq \mathsf{AQSMSL}$ and any $a, b \in A$ such that $a \in \mathcal{F}^{\mathfrak{A}}$ but $(\neg^{\mathfrak{A}} b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b) \not\leqslant^{\mathfrak{A}} b$, in which case, by (2.8) and Corollary 4.7, there are some $\mathfrak{B} \in (\mathsf{QSMSL} \cup \{\mathfrak{K}_2\})$ and some $g \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $(\neg^{\mathfrak{B}} g(b) \wedge^{\mathfrak{B}} \neg^{\mathfrak{B}}\neg^{\mathfrak{B}} g(b)) \not\leqslant^{\mathfrak{B}} g(b)$, and so $\mathfrak{B} = \mathfrak{K}_2$ as well as $g(b) = 0$. Likewise, by Lemmas 5.1 and 5.22, there is some $e \in \hom(\mathfrak{A}, \mathfrak{B}_2) \neq \varnothing$. Let $\mathsf{K} \triangleq ((\mathsf{QSMSL} \cap \mathbf{S}_{>1}\mathfrak{MS}_6))$, in which case $\mathcal{H} \triangleq \hom(\mathfrak{A}, \mathsf{K}) \ni h \triangleq (e \circ \epsilon^3_2 \circ \epsilon^5_3)$ is non-empty as well as finite, for both K and all its members are so, in view of the finiteness of $MS_6 = 2^3$, and so there is a bijection $\bar{h}$ from $n \triangleq |\mathcal{H}| \in (\omega \setminus 1)$ onto $\mathcal{H}$ such that $h_0 = h$. By contradiction, we prove that $\mathcal{A} \triangleq (\prod_{j \in n}((g^{-1}[1] \cap h_j^{-1}[\mathcal{F}^{\mathfrak{MS}_6}_{\mathrm{W}}]) \cup h_j^{-1}[\Im^{\mathfrak{MS}_6}_{\neg}])) = \varnothing$. For suppose there is some $\bar{c} \in \mathcal{A} \subseteq A^n$. Let $d \triangleq \boldsymbol{\iota}^{\mathfrak{A}}_{x_0,n}(\bar{c})$, in which case, as $\Im^{\mathfrak{MS}_6}_{\neg} = (2^3 \setminus (\mathrm{img}\, \epsilon^5_3))$ is disjoint with $(\mathrm{img}\, \epsilon^5_3) \supseteq (\mathrm{img}\, h_{[0]})$, $g(c_0) = 0$, and so, by induction on any $k \in ((n+1) \setminus 1) \ni n$, we see that $g(\boldsymbol{\iota}^{\mathfrak{A}}_{x_0,k}(\bar{c}\restriction k)) = 0$. In particular, when taking $k = n$, we conclude that $g(d) = 0$, in which case $d \notin \mathcal{F}^{\mathfrak{A}}$, for $g(\neg^{\mathfrak{A}} b) = 1 \not\leqslant 0$, and so, since $\mathfrak{A} \models (4.13)[x_0/d]$, $d \notin \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}$. On the other hand, for every $l \in n$, $h_l(c_l) \in (\mathcal{F}^{\mathfrak{MS}_6}_{\mathrm{W}} \cup \Im^{\mathfrak{MS}_6}_{\neg}) = \mathcal{F}^{\mathfrak{MS}_6}_{\mathrm{W}}$, in which case, by the left alternative of Claim 5.2, $h_l(d) \in \mathcal{F}^{\mathfrak{MS}_6}_{\mathrm{W}}$, and so, by the total negative idempotence of $\mathfrak{K}_2$, (2.8) and Corollary 4.7, $d \in \mathcal{F}^{\mathfrak{A}}_{\mathrm{W}}$. This contradiction shows that $\mathcal{A} = \varnothing$, in which case there is some $m \in n$ such that $h_m[g^{-1}[1]|A]$ is disjoint with $\mathcal{F}^{\mathfrak{MS}_6}_{\mathrm{W}}|\Im^{\mathfrak{MS}_6}_{\neg}$, i.e., is subsumed by $\{\langle 0,0,0\rangle\}|(\mathrm{img}\, \epsilon^5_3)$, and so, by (4.21), $f \triangleq (h_m \circ (\epsilon^5_3)^{-1} \circ \chi^{3\setminus 1}_3) \in \hom(\mathfrak{A}, \mathfrak{B}_2)$, while $f[g^{-1}[1]] \subseteq 1$. Thus, $f(a\|b) = (1\|0)$, in which case $f(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) = 1 \not\leqslant 0 = f(b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$, and so $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \not\leqslant^{\mathfrak{A}} (b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$. Then, Corollaries 4.7, 5.4, (5.15), Lemmas 5.7, 5.22, Theorem 5.24 and the locality of quasi-varieties complete the argument. □

5.4. **Embaddibility lemmas.**

Lemma 5.26. $\mathfrak{K}_{3\{[\langle +1:1\rangle]\}} \times \mathfrak{B}_2$ *is embedable into any* $\mathfrak{A} \in ((\mathsf{NI}(\mathsf{A})\{[\mathsf{Q}]\mathsf{S}\}\mathsf{K}\langle \mathsf{M} \rangle\{\mathsf{S}\}\mathsf{L}(\cup\mathsf{TNIMSL})\langle \cup(\mathsf{A})\mathsf{DML}\rangle) \setminus (\mathsf{W})\langle \mathsf{M} \rangle \mathsf{RMSL})$.

Proof. Then, there are some $a, b\langle, c\rangle \in A$ such that $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a$, $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} b)$ but $(\neg^{\mathfrak{A}} b\langle \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} c\rangle) \not\leqslant^{\mathfrak{A}} ((\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}) b\langle \vee^{\mathfrak{A}} c\rangle)$, in which case, by (4.2), (4.5)

and (4.6), we have $((d|e)\{[\langle\|f\rangle]\}) \triangleq (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)\{[\langle\|(c \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}c \vee^{\mathfrak{A}} d\rangle]\}) = \{[\langle\| \geqslant^{\mathfrak{A}} \rangle]\}(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(d|e)\{[\langle\|(\neg^{\mathfrak{A}}f/d)\rangle]\})(\geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}d|\,\|)$, while, applying (4.3) twice, by (4.1) and (4.5), we get $(d\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}e) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}}d\vee^{\mathfrak{A}}e)$, whereas, by (2.8) and Corollary 5.4 (and 5.5) ⟨as well as 5.8⟩, there are some $\mathfrak{C} \in \{\mathfrak{K}_{3\{[+1:1]\}} \times \mathfrak{B}_2\{,\mathfrak{S}_3\}\langle,\mathfrak{DM}_4\rangle(,\mathfrak{K}_2)\}$ and some $h \in \hom(\mathfrak{A},\mathfrak{C})$ such that $(\neg^{\mathfrak{C}}h(b)\langle\wedge^{\mathfrak{C}}\neg^{\mathfrak{C}}\neg^{\mathfrak{C}}h(c)\rangle) \not\leqslant^{\mathfrak{C}} ((\neg^{\mathfrak{C}}\neg^{\mathfrak{C}})h(b)\langle\vee^{\mathfrak{C}}h(c)\rangle)$, and so $\mathfrak{C} \triangleq (\mathfrak{K}_{3\{[\langle+1:1\rangle]\}} \times \mathfrak{B}_2)$ and $h((a\|d)|(b\|e)\{[\langle|f\rangle]\}) = \langle 1|0\{[\langle|2\rangle]\},1\rangle$, for $\neg^{\mathfrak{C}}h(a) \leqslant^{\mathfrak{C}} h(a)$ and $(h(a) \wedge^{\mathfrak{C}} \neg^{\mathfrak{C}}h(b)) \leqslant^{\mathfrak{C}} (\neg^{\mathfrak{C}}h(a) \vee^{\mathfrak{C}} h(b))$. In that case, using (4.1), (4.2), (4.5) and (4.6), it is routine checking that the mapping $g : ((3\{[\langle+1\rangle]\}) \times 2) \to A$, given by:

$$\begin{array}{rcl} g(\langle 0|1,1\rangle) & \triangleq & ((d \wedge^{\mathfrak{A}} (e \vee^{\mathfrak{A}} (e|\neg^{\mathfrak{A}}d)))\{[\langle\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}f\rangle]\}), \\ g(\langle(2\{[\langle+1\rangle]\})|1,0\rangle) & \triangleq & \neg^{\mathfrak{A}}g(\langle 0|1,1\rangle), \\ g(\langle 0|(2\{[\langle+1\rangle]\}),0|1\rangle) & \triangleq & (g(\langle 0,1\rangle)(\wedge|\vee)^{\mathfrak{A}}g(\langle 2\{[+1]\},0\rangle)), \\ \{[\langle g(\langle 2,1\rangle) & \triangleq & (((d \wedge^{\mathfrak{A}} e) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(d \wedge^{\mathfrak{A}} e)) \wedge^{\mathfrak{A}} f),\rangle]\} \end{array}$$

is a homomorphism from $\mathfrak{K}_{3\{[\langle+1:1\rangle]\}} \times \mathfrak{B}_2$ to $\mathfrak{A}$ such that $(g \circ h) = \Delta_{(3\{[\langle+1\rangle]\})\times 2}$, and so it is injective, as required. □

Lemma 5.27. *$\mathfrak{K}_4$ is embeddable into any $\mathfrak{A} \in (\mathsf{NIQSMSL} \setminus \mathsf{ASL}) \supseteq (\mathsf{RQSKSL} \setminus \mathsf{SL}) \supseteq (\mathsf{RKL} \setminus \mathsf{BL})$.*

Proof. Then, by (4.2), there are some $a, b \in A$ such that $c \triangleq (\neg^{\mathfrak{A}}a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a) \neq d \triangleq (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \vee^{\mathfrak{A}} c) \geqslant^{\mathfrak{A}} c$, in which case, by (4.1), (4.3), (4.5) and (4.6), we have both $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d = d$ and $\neg^{\mathfrak{A}}d \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}c \leqslant^{\mathfrak{A}} c = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c \neq \neg^{\mathfrak{A}}c$, for $\mathfrak{A}$ is neither idempotent nor one-element, and so $\neg^{\mathfrak{A}}c \neq \neg^{\mathfrak{A}}d$. In this way, $\{\langle 0, \neg^{\mathfrak{A}}d\rangle, \langle 1, \neg^{\mathfrak{A}}c\rangle, \langle 2, c\rangle, \langle 3, d\rangle\}$ is an embedding of $\mathfrak{K}_4$ into $\mathfrak{A}$. Finally, Corollary 5.11 completes the argument. □

Lemma 5.28. *$\mathfrak{DM}_4$ is embeddable into any $\mathfrak{A} \in (\mathsf{SMSL} \setminus (\mathsf{NISMSL} \cup \mathsf{WKSL})) \supseteq (\mathsf{AQSMSL} \setminus (\mathsf{NIAQSMSL} \cup \mathsf{AQSKSL}))$.*

Proof. In that case, by (4.2) and Corollary 5.6, there are some $a, b \in A$ such that $\mathfrak{A} \not\models \{(5.5)\}[x_0/\neg^{\mathfrak{A}}a, x_1/b]$, and so (5.5) is not true in the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{\neg^{\mathfrak{A}}a, \neg^{\mathfrak{A}}b\}$ under $[x_0/\neg^{\mathfrak{A}}a, x_1/\neg^{\mathfrak{A}}b]$. On the other hand, by (4.1), (4.5), (4.6) and induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we have $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\neg^{\mathfrak{A}}a, \neg^{\mathfrak{A}}b) = \varphi^{\mathfrak{A}}(\neg^{\mathfrak{A}}a, \neg^{\mathfrak{A}}b)$, in which case $\mathfrak{B}$ is a De Morgan lattice, and so $\mathfrak{DM}_4$, being embedable into $\mathfrak{B}$, in view of [18, Case 8 of Proof of Theorem 4.8], is so into $\mathfrak{A}$, as required, because, by Corollary 4.7, $\mathsf{AQWKSL} = \mathsf{AQSKSL}$. □

Lemma 5.29. *$\mathfrak{DM}_4 \times \mathfrak{B}_2$ is embeddable into an arbitrary $\mathfrak{A} \in (\mathsf{NIMSL} \setminus \mathsf{WKSL}) \supseteq (\mathsf{NIAQSMSL} \setminus \mathsf{AQSKSL})$.*

Proof. Then, there are some $a, b \in A$ such that, by (4.2), $c \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a) \not\leqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}}b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b)$, in which case, by (4.1), (4.5) and (4.6), we have both $\neg^{\mathfrak{A}}(c|d)(\geqslant | \leqslant)^{\mathfrak{A}}(c|d) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$, and so, by induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c,d) = \varphi^{\mathfrak{A}}(c,d)$. Thus, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent De Morgan lattice such that $\mathfrak{B} \not\models \mathcal{K}[x_0/c, x_1/d]$, in which case $\mathfrak{DM}_4 \times \mathfrak{B}_2$ being embeddable into $\mathfrak{B}$, in view of the proof of [18, Lemma 4.10], is so into $\mathfrak{A}$, as required, because, by Corollary 4.7, $\mathsf{AQWKSL} = \mathsf{AQSKSL}$. □

Lemma 5.30. *Let $\mathfrak{A} \in ([\mathsf{PS}]\mathsf{MSL} \setminus (\mathsf{NIMSL} \cup \mathsf{NDML}))$. Then, $\{\mathfrak{K}_{(4[+1]):i} \mid i \in (2[-1])\})) \cap \mathbf{IS}\mathfrak{A}) \neq \varnothing$. In particular, $\mathfrak{K}_{4:(0|1)}$ is embeddable into any member of $((\mathsf{P}|(\{\mathsf{A}\}\mathsf{Q}))\mathsf{SMSL} \setminus (\mathsf{NIMSL} \cup (\mathsf{N}|\{\mathsf{N}\|\mathsf{A}\})\mathsf{DML}))$.*

Proof. Then, by (4.1), (4.5) and (4.6), there are some $a, e \in A$ such that $\neg^{\mathfrak{A}}e = e$ and $c \neq d \neq b$, where $b, c, d \in A$ are as in Lemma 5.23, in which case $b \leqslant^{\mathfrak{A}} (f|g) \triangleq ((e \wedge^{\mathfrak{A}} (c|d)) \vee^{\mathfrak{A}} b) = (g \wedge^{\mathfrak{A}} (c|d))$, and so, by (4.1) and (4.5), we have

$b \neq f \leqslant^{\mathfrak{A}} g = \neg^{\mathfrak{A}}(f|g) \notin \{c,d\}$, for, otherwise, we would get $b = g = d$. Consider the following complementary cases:

- $g \leqslant^{\mathfrak{A}} c$,
  in which case $\{\langle 0,b\rangle, \langle 1,g\rangle, \langle 2,c\rangle, \langle 3,d\rangle\}$ is an embedding of $\mathfrak{K}_{4:1}$ into $\mathfrak{A}$, and so the latter is not pseudo-strong, for the former is not so, in view of Lemma 4.6.
- $g \not\leqslant^{\mathfrak{A}} c$,
  in which case, by (4.3), we have $c \not\leqslant^{\mathfrak{A}} g$, and so $\{f, g, c, g \vee^{\mathfrak{A}} c\}$ is a non-degenerated diamond of $\mathfrak{A}{\restriction}\Sigma_+$. Then, [as $(g \vee^{\mathfrak{A}} c) = d$, for $\mathfrak{A} \models (4.12)[x_0/c, x_1/e]$] by (4.1) and (4.5),
  $$\{\langle 0,0,0,b\rangle, \langle 0,1,0,f\rangle, [\langle 0,1,1,c\rangle,] \langle 1,1,0,g\rangle, \langle 1,1,1,d\rangle\}$$
  is an embedding of $\mathfrak{MS}_{(4[+1]):0}$ into $\mathfrak{A}$.

In this way, (4.18) and Lemma 4.6 complete the argument. □

Lemma 5.31. *$\mathfrak{K}_{5:1}$ is embeddable into an arbitrary $\mathfrak{A} \in ((\mathsf{NI(A)QSMSL} \cup \mathsf{(W)MR(A)QSMSL}) \setminus \mathsf{PSMSL})$.*

Proof. Take any $a, e \in A$ such that $\mathfrak{A} \not\models (4.12)[x_0/a, x_1/e]$, in which case $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \not\leqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} f)$, where $f \triangleq (\neg^{\mathfrak{A}}e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e) \geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}f$, in view of (4.5), and so $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \neq a$. On the other hand, by Lemmas 4.6, 5.7, Corollary 5.4 and Theorem 5.12, $\mathsf{NI(A)QSMSL} \cup \mathsf{(W)MR(A)QSMSL}$ is the pre-variety generated by $\mathsf{K} \triangleq \{\mathfrak{K}_{4:1} \times \mathfrak{B}_2, \mathfrak{DM}_4(,\mathfrak{K}_2)\}$, in which case, by (2.8), there are some $\mathfrak{C} \in \mathsf{K}$ and $h \in \hom(\mathfrak{A}, \mathfrak{C})$ such that $h(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a) \not\leqslant^{\mathfrak{C}} h(a \vee^{\mathfrak{A}} f)$, and so $\mathfrak{C} = (\mathfrak{K}_{4:1} \times \mathfrak{B}_2)$, while $\pi_1(h(f)) = 1$, whereas $\pi_0(h((a|e)) = (2|1)$. Let $b, c, d \in A$ be as in Lemma 5.23 and $g \triangleq \{\langle 0,0,b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}f\rangle, \langle 1,0,\neg^{\mathfrak{A}}f\rangle, \langle 1,1,f\rangle, \langle 2,1,c \vee^{\mathfrak{A}} f\rangle, \langle 3,1,d \vee^{\mathfrak{A}} f\rangle\} : \Re^{\mathfrak{K}_{4:1}} \to A$, in which case, for all $\bar{\imath}, \bar{\jmath} \in \Re^{\mathfrak{K}_{4:1}}$, $(\bar{\imath} \leqslant^{\mathfrak{D}_4 \times \mathfrak{D}_2} \bar{\jmath}) \Rightarrow (g(\bar{\imath}) \leqslant^{\mathfrak{A}} g(\bar{\jmath}))$ as well as $h(g(\bar{\imath})) = \bar{\imath}$, and so, since $\Re(\mathfrak{K}_{4:1}){\restriction}\Sigma_+$ is a chain lattice, by (4.1), (4.5) and (4.6), g is an embedding of $\Re(\mathfrak{K}_{4:1}) \in \mathbf{I}(\mathfrak{K}_{5:1})$ into $\mathfrak{A}$, as required. □

Lemma 5.32. *$\mathfrak{K}_2$ is embeddable into any $\mathfrak{A} \in \mathsf{MSL}$ not satisfying* (5.7).

Proof. Take any $a \in (\Im^{\mathfrak{A}}_{\neg} \setminus \Im^{\mathfrak{A}}) \neq \varnothing$, in which case, by (4.2), $a \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}a$, and so $\{\langle i, (\neg^i)^{\mathfrak{A}}a\rangle \mid i \in 2\}$ is an embedding of $\mathfrak{K}_2$ into $\mathfrak{A}$, as required. □

This, by (5.9), immediately yields:

Corollary 5.33. *$\mathfrak{K}_2$ is embeddable into any $\mathfrak{A} \in (\mathsf{TNIMSL} \setminus \mathsf{OMSL})$.*

This, in its turn, by Corollary 4.7, immediately yields:

Corollary 5.34. TNIMSL *has no proper non-trivial sub-/pre-/quasi-/variety.*

Lemma 5.35. *$\mathfrak{K}_2 \times \mathfrak{B}_2$ is embeddable into any $\mathfrak{A} \in (\mathsf{NIMSL} \setminus \mathrm{Mod}(((4.13)))) \supseteq (\mathsf{NINDML} \setminus \mathrm{Mod}((4.13))) \supseteq (\mathsf{NIADML} \setminus \mathsf{VDML})$.*

Proof. Then, there is some $a \in A$ such that $(a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}a) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a$ but $\neg^{\mathfrak{A}}a \not\leqslant^{\mathfrak{A}} a$, in which case $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \notin \{a, \neg^{\mathfrak{A}}a\}$, and so $a \not\leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}a$, Thus, by (4.1) and (4.6), $\{\langle 0,1,a\rangle, \langle 1,0,\neg^{\mathfrak{A}}a\rangle, \langle 0,0,a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a\rangle, \langle 1,1,\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a\rangle\}$ is an embedding of $\mathfrak{K}_2 \times \mathfrak{B}_2$ into $\mathfrak{A}$, as required, in view of Corollary 4.7, due to which $\mathsf{ADML} \subseteq \mathsf{NDML}$. □

Lemma 5.36. *$\mathfrak{B}_2 \oplus 1$ (resp., $\mathfrak{K}_3 \oplus 1$) is embedable into any $\mathfrak{A} \in (((\mathsf{I})\mathsf{MSL} \setminus \mathsf{QSMSL}) \cap \mathrm{Mod}((5.7)))$.*

Proof. Then, we have some $a(, b) \in A$ such that ($\neg^{\mathfrak{A}}b = b$ but) $c \triangleq (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a) \neq d \triangleq (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a)$, in which case, by (4.1), (4.2) and (4.6), $c \leqslant d = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c \leqslant^{\mathfrak{A}} (e \triangleq ((b \vee^{\mathfrak{A}} d) \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}c) = \neg^{\mathfrak{A}}e) \leqslant^{\mathfrak{A}} f \triangleq \neg^{\mathfrak{A}}c = \neg^{\mathfrak{A}}d$, and so $c \neq f$, for $c \neq d$. Hence, as $\mathfrak{A} \models (5.7)[x_0/c]$, $d \neq f$ (in which case $e \notin \{d, f\}$), and so

$\{\langle 0,0,c\rangle, \langle 1,0,d\rangle, (\langle 1,1,e\rangle,)\langle 1,1(+1),f\rangle\}$ is an embedding of $\mathfrak{B}_2 \oplus 1$ (resp., $\mathfrak{K}_3 \oplus 1$) into $\mathfrak{A}$, as required. □

On the other hand, both $\mathfrak{K}_2$ and $\mathfrak{B}_2 \oplus 1$, being finite, are expandable to bounded MS lattices, in which case neither of these is an MS algebra, and so, by Lemmas 5.32 and 5.36, we eventually conclude that QSMSL is the infinitary universal unbounded approximation of MSA in the following sense:

Corollary 5.37. QSMSL *is the greatest infinitary universal (viz., abstract hereditary) subclass of* MSL *disjoint with* $(\mathsf{BMSL} \setminus \mathsf{MSA})\restriction\Sigma^-_+$.

Nevertheless, the stipulation "infinitary universal"/"abstract hereditary" can be neither omitted nor replaced by the one "image-closed [ultra-]multiplicative", simply because, due to the truth of (4.7) in bounded MS lattices, $\mathsf{MSA}\restriction\Sigma^-_+$ is the subclass of MSL, relatively axiomatized by the single first-order positive Σ^-_+-equational $\exists\forall$-sentence $\exists x_0 \forall x_1((x_1 \lessapprox x_0)\&(\neg x_0 \lessapprox x_1))$, in which case it is image-closed (in particular, abstract) and [ultra-]multiplicative but is not hereditary, for $(\mathsf{MSA}\restriction\Sigma^-_+) \not\ni \mathfrak{MS}_2 \preceq \mathfrak{MS}_6 \in (\mathsf{MSA}\restriction\Sigma^-_+)$, and so, by Lemma 2.9, is not submultiplicative.

Lemma 5.38. $\mathfrak{MS}_{5:1}$ *is embedable into an arbitrary* $\mathfrak{A} \in ((\mathsf{MSL} \cap \mathrm{Mod}(((4.13)))) \setminus \mathrm{Mod}((4.13))) \supseteq (\mathsf{NQSRPVQSKSL} \setminus [\mathsf{N}]\mathsf{QSRVQSKSL})$.

Proof. Then, there is some $a \in (\mathcal{F}^{\mathfrak{A}}_{\mathrm{W}} \setminus \mathcal{F}^{\mathfrak{A}})$ such that $(a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \neq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a$, in which case, by (4.2) and (4.3), $a \not\leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} a$, and so, by (4.1), (4.2), (4.5) and (4.6), $\{\langle 0,0,a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a\rangle, \langle 0,1,a\rangle, \langle 1,0,\neg^{\mathfrak{A}} a\rangle, \langle 1,1,a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a\rangle, \langle 1,2,\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a\rangle\}$ is an embedding of $\mathfrak{MS}_{5:1}$ into $\mathfrak{A}$, as required, in view of Theorems 5.24 and 5.25. □

Lemma 5.39. $\mathfrak{B} \triangleq ((\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2)$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NIPVQSKSL} \setminus \mathsf{NQSRMSL}) \supseteq (\mathsf{NIVQSKSL} \setminus [\mathsf{N}]\mathsf{QSRVQSMSL})$.

Proof. Then, there are some $a, b \in A$ such that $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a$, $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} b)$ and $(c \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} c) = (f \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} c)$, where $c \triangleq (a \wedge^{\mathfrak{A}} b)$ and $(d|e|f) \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b|c)$, but $(\neg^{\mathfrak{A}} b \wedge^{\mathfrak{A}} e) \not\leqslant^{\mathfrak{A}} b$, in which case, by (2.8) and Corollary 5.19, there are some $\mathfrak{C} \in ((\{\mathfrak{K}_{4:1}, \mathfrak{K}_3 \oplus 1\} \otimes \{\mathfrak{B}_2\}) \cup \{\mathfrak{MS}_{5:1}\})$ and some $h \in \hom(\mathfrak{A}, \mathfrak{C})$ such that $(\neg^{\mathfrak{C}} h(b) \wedge^{\mathfrak{A}} h(e)) \not\leqslant^{\mathfrak{C}} h(b)$, and so, since $\neg^{\mathfrak{C}} h(a) \leqslant^{\mathfrak{C}} h(a)$, $(h(a) \wedge^{\mathfrak{C}} \neg^{\mathfrak{C}} h(b)) \leqslant^{\mathfrak{C}} (\neg^{\mathfrak{C}} h(a) \vee^{\mathfrak{C}} h(b))$ and $(h(c) \vee^{\mathfrak{C}} \neg^{\mathfrak{C}} h(c)) = (h(f) \vee^{\mathfrak{C}} \neg^{\mathfrak{C}} h(c))$, by the quasi-strength of $\mathfrak{K}_{4:1}$ and the nearly quasi-strong regularity of $\mathfrak{MS}_{5:1}$ (cf. Theorem 5.24), $\mathfrak{C} = \mathfrak{B}$, while, by the truth of (4.13) in $\mathfrak{B}$, $h((a|d)\|(b|e)\|(c|f)) = \langle 1\|(0|1)|(0|1), 1\|0\|0, 1\rangle$, because $\neg^{\mathfrak{B}} h(b) \not\leqslant^{\mathfrak{B}} h(e)$, for $\neg^{\mathfrak{B}} h(b) \not\leqslant h(b)$. In that case, using (4.1), (4.2), (4.3), (4.5) and (4.6), it is routine checking that the mapping $g : B \to A$, given by:

$$\begin{aligned}
g(0,0,0) &\triangleq (c \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c),\\
g(0,0,1) &\triangleq c,\\
g(1,0,0) &\triangleq (f \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} f),\\
g(1,0,1) &\triangleq f,\\
g(1,2,1) &\triangleq (f \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} f),\\
g(1,2,0) &\triangleq \neg^{\mathfrak{A}} f,\\
g(1,1,1) &\triangleq ((e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} d) \wedge^{\mathfrak{A}} d),\\
g(1,1,0) &\triangleq ((d \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} d),
\end{aligned}$$

is a homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ such that $(g \circ h) = \Delta_B$, and so is injective, as required, in view of Theorem 5.25. □

5.5. **Relative subdirect irreducibility of non-trivial subalgebras of generating algebras.** First, by Footnote 1, (4.21), (4.22), (5.13), Lemma 2.14 and the simplicity of $(\mathfrak{B}|\mathfrak{K})_{(2|3)[,01]}$ (cf., e.g., Remark 4.3), we immediately have:

Lemma 5.40. *Congruences of* $\mathfrak{A} \triangleq (\mathfrak{S}|\mathfrak{K})_{3|(4:(0\|1))[,01]}$ *form the three-element chain* $\Delta_{3|4} \subsetneq (\ker h) \subsetneq (3|4)^2$*, where* $h \in \hom^{\mathrm{S}}(\mathfrak{A}, (\mathfrak{B}|\mathfrak{K})_{(2|3)[,01]}) \neq \varnothing$.

Lemma 5.41. *Let* $\mathfrak{A} \in (\mathsf{MSL} \setminus \mathsf{OMSL})$*,* $\mathfrak{B} \in \{\mathfrak{B}_2, \mathfrak{S}_3\}$ *and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$*. Then,* $\mathfrak{B}_2 \preceq h[\mathfrak{A}] \preceq \mathfrak{A}$.

Proof. In that case, $h[\mathfrak{A}] \in \mathbf{S}\mathfrak{B} \subseteq (\{\mathfrak{S}_3\} \cup \mathbf{I}(\mathfrak{B}_2))$, and so [unless $|A| > 2$], by (4.21), Corollary 5.1, Lemmas 4.6, 5.3, 5.15 and (5.9), $\mathfrak{B}_2 \preceq h[\mathfrak{A}][\in \mathbf{I}\mathfrak{B}_2 \subseteq \mathbf{IS}\mathfrak{A}]$. [Otherwise, by Lemma 4.6, $\mathbf{H}\mathfrak{A} \ni h[\mathfrak{A}] = \mathfrak{S}_3 \notin \mathsf{NDML}$, in which case $\mathfrak{A} \notin \mathsf{NDML}$, and so Lemma 5.23 completes the argument.] □

Let $\mathfrak{K}_1 \triangleq (\mathfrak{K}_2{\restriction}\{1\}) \in \mathsf{OMSL}$.

Theorem 5.42. *Let* $\mathfrak{C} \in (\{\mathfrak{B}_2\}[\cup(\{\mathfrak{S}_3\}(\cap\varnothing))]\{\cup(\{\mathfrak{K}_3\}[\cap\varnothing])\})$*,* $\mathsf{P} \subseteq \mathsf{MSL}$ *a pre-variety,* $\mathfrak{F} \in (\mathsf{P} \cap (\mathbf{S}_{>1}(([([(\{\mathfrak{K}_2\}(\cap\varnothing))(\cup\{\mathfrak{K}_3 \oplus 1\})) \cup](\mathbf{I}\mathsf{MS} \cap \mathsf{AQSMSL}\{\cap\{\mathfrak{K}_2\}\}[\cap\varnothing]))))[\setminus \mathbf{IS}\{\mathfrak{K}_2 \times \mathfrak{B}_2(, \mathfrak{K}_3 \times \mathfrak{B}_2, \mathfrak{K}_3 \oplus 1)\}]))$ *and [both]* $\mathfrak{K} \triangleq (\pi_0^2{\restriction}F)[\mathfrak{F}]$*[,* $g \triangleq (\chi_3^{3\setminus 1}{\restriction}C)$ *and* $\mathfrak{G} \triangleq ((\pi_0^{2(+1)}{\restriction}F) \odot ((\pi_{1(+1)}^{2(+1)}{\restriction}F) \circ g))[\mathfrak{F}]$ *(as well as* $\mathfrak{H} \triangleq ((\pi_0^3{\restriction}F) \odot (\pi_1^3{\restriction}F))[\mathfrak{F}]$*); cf.* (2.7)*].* {*Suppose* $\mathfrak{F} \in \mathrm{Mod}((4.13))$}*. Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftarrow | \Rightarrow ((((((\{\mathfrak{K}\}[\cap\varnothing])[\cup\{\mathfrak{G}(, \mathfrak{H})\}]) \cap \mathsf{P}) \neq \varnothing) \Rightarrow (\mathfrak{F} \preceq ((\{\mathfrak{K}\}[\cap\varnothing]) \cup \{\mathfrak{C}\})))$.

Proof. In that case, by (4.21), Corollary 5.4, Lemma 5.15 and (5.9), $\mathfrak{F}$, being non-one-element, is not totally negatively idempotent, and so, by Lemma 5.3, $\mathfrak{B}_2$, being embedable into $\mathfrak{F}$, belongs to P [while, in the ()-non-optional case, $\mathfrak{F} \notin \mathsf{TNIMSL} \supseteq \mathbf{IS}\mathfrak{K}_2 \supseteq \mathbf{IS}\mathfrak{K}$].

We start from proving the right part. For suppose $((((\{\mathfrak{K}\}[\cap\varnothing])[\cup\{\mathfrak{G}(, \mathfrak{H})\}])\cap\mathsf{P}) \neq \varnothing$ but $\mathfrak{F} \not\preceq ((\{\mathfrak{K}\}[\cap\varnothing])\cup\{\mathfrak{C}\})$, in which case $\mathfrak{F} \not\preceq \{\mathfrak{K}, \mathfrak{C}\}$ [(for $\mathfrak{K} \in \mathbf{S}(\mathfrak{K}_3 \oplus 1)$)], and so, since $(e/f) \triangleq (\pi_{0/1}^2{\restriction}F) \in \hom^{\mathrm{S}}(\mathfrak{F}, \mathfrak{K}/\mathfrak{C})$ is then not injective, for, otherwise, it would be an embedding of $\mathfrak{F}$ into $\mathfrak{K}/\mathfrak{C}$, $(\theta/\vartheta) \triangleq \ker(e/f) \neq \Delta_F = (\theta \cap \vartheta)$, while, by (2.5) and Lemma 5.41, $\vartheta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$ {unless both $\mathfrak{C} = \mathfrak{K}_3$ and $1 \in \pi_1^2[F]$}. {Otherwise, $\mathfrak{F} \in \mathbf{S}(\mathfrak{K}_2 \times \mathfrak{K}_3\}$, in which case $\mathfrak{K} \in \mathbf{S}\mathfrak{K}_2 = \{\mathfrak{K}_1, \mathfrak{K}_2\}$, and so there are some $a, b \in F$ such that $\pi_{01}^2(a \wr b) = 1$. Then, by (5.2) and (5.3), $c \triangleq \neg^{\mathfrak{F}}\boldsymbol{\pi}^{\mathfrak{F}}(\boldsymbol{\mu}^{\mathfrak{F}}(a), \boldsymbol{\mu}^{\mathfrak{F}}(b)) \in \Im^{\mathfrak{F}}$, in which case, as $\mathfrak{F}$, being non-one-element, is idempotent but not totally negatively idempotent, by Lemma 5.3, $\mathfrak{C} \in \mathbf{IS}\mathfrak{F} \subseteq \mathsf{P}$, and so, by (2.5), $\vartheta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$.} Thus, anyway, $\vartheta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$. In particular, $\mathfrak{F} \notin \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$, whenever $\mathfrak{K} \in \mathsf{P}$, because, in that case, by (2.5), $\theta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$.

[Conversely, assume $\mathfrak{K} \notin \mathsf{P}$, in which case $(\{\mathfrak{G}(, \mathfrak{H})\} \cap \mathsf{P}) \neq \varnothing$, and so $\mathfrak{G} \in \mathsf{P}$ (for $\mathfrak{H} = \mathfrak{K} \notin \mathsf{P}$). (And what is more, $\mathfrak{F} \in \mathbf{S}((\mathfrak{K}_2 \times \mathfrak{K}_3) \times \mathfrak{B}_2) \ni \mathfrak{I} \triangleq ((\mathfrak{K}_1 \times \mathfrak{K}_3) \times \mathfrak{B}_2) \in \mathbf{I}(\mathfrak{K}_3 \times \mathfrak{B}_2)$, for $\mathfrak{K}_1 \in \mathbf{S}\mathfrak{K}_2$ is one-element, in which case $\mathfrak{J} \triangleq (\mathfrak{F}{\restriction}(F \cap I)) \in (\mathbf{S}(\mathfrak{F}) \cap \mathbf{S}(\mathfrak{I})) \subseteq (\mathsf{P} \cap \mathbf{IS}(\mathfrak{K}_3 \times \mathfrak{B}_2))$, and so, since, by (2.7), $f' \triangleq (((F \times \{1\}) \odot (\pi_1^3{\restriction}F)) \odot (\pi_2^3{\restriction}F)) \in \hom(\mathfrak{F}, \mathfrak{J})$ is then non-injective, by (2.5), we conclude that $\Delta_F \neq \vartheta' \triangleq (\ker f') \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$.) Then, by (4.21), $\mathbf{S}(\mathfrak{K}_2 \times \mathfrak{B}_2) \ni \mathfrak{G}$ is disjoint with $\mathbf{I}\mathfrak{F}$, in which case $e' \triangleq ((\pi_0^{2(+1)}{\restriction}F) \odot ((\pi_{1(+1)}^{2(+1)}{\restriction}F) \circ g)) \in \hom^{\mathrm{S}}(\mathfrak{F}, \mathfrak{G})$ is not injective, and so, by (2.5), $\Delta_F \neq \theta' \triangleq (\ker e') \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{F})$. Hence, $\mathfrak{F} \notin \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$, because, by (2.1), $\Delta_F \subseteq (\theta' \cap \vartheta(')) \subseteq (\bigcap_{i \in (2(+1))} \ker(\pi_i^{2(+1)}{\restriction}F)) = \Delta_E$.]

Next, if $\mathfrak{F} \preceq ((\{\mathfrak{K}\}[\cap\varnothing]) \cup \{\mathfrak{C}\})$, then, by Theorem 4.4, $\mathfrak{F} \in (\mathsf{P} \cap \mathbf{IS}_{>1}\mathsf{MS}\}) \subseteq (\mathbf{I}(\mathsf{MS} \cap \mathsf{P}) = \mathrm{SI}(\mathsf{P}) \subseteq \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$.

Now, assume $\mathfrak{F} \not\preceq ((\{\mathfrak{K}\}[\cap\varnothing]) \cup \{\mathfrak{C}\})$ but $((((\{\mathfrak{K}\}[\cap\varnothing])[\cup\{\mathfrak{G}(, \mathfrak{H})\}]) \cap \mathsf{P}) = \varnothing$. Put $\mathfrak{A} \triangleq (\pi_0^{2[(+1)]}{\restriction}F)[\mathfrak{F}]$, both $\mathfrak{B} \triangleq (\mathfrak{K}_3 \| \mathfrak{B}_2 \| \mathfrak{K}_1) \in \mathsf{KL}$ and $h \triangleq (((\chi_A^{A \setminus \mathfrak{I}^{\mathfrak{A}}} \odot \chi_A^{\mathfrak{F}^{\mathfrak{A}}}) \circ (\epsilon_{3:0}^4)^{-1}) \| \chi_A^{\mathfrak{F}^{\mathfrak{A}}} \| (A \times \{1\})) \in \hom(\mathfrak{A}, \mathfrak{B})$, whenever $\mathfrak{A} \in (\mathbf{I}\{\mathfrak{K}_{4:1}\} \| \mathbf{I}\{\mathfrak{S}_3\} \| (\mathsf{MSL} \setminus$

$\mathbf{I}\{\mathfrak{K}_{4:1}, \mathfrak{S}_3\}))$ [as well as [(both $\mathfrak{E} \triangleq \mathfrak{K}_3$ and) $\mathfrak{D} \triangleq \mathfrak{B}_2$], in which case, by (2.7), (2.5), (4.21), (5.13), Lemmas 4.6 and 5.40, both $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, A^2, \ker h\}$ and $\mathrm{Co}(\mathfrak{C}) = \{\Delta_C, C^2[, \ker g]\}$ [while $g \in \hom_{(\mathrm{I})}(\mathfrak{C}, \mathfrak{D})$ (whereas $\mathrm{Co}(\mathfrak{E}) = \{\Delta_E, E^2\}$)]. And what is more, $\mathfrak{A} \notin \mathsf{P}$ [because, otherwise, by (4.21), $\mathfrak{G} \in \mathbf{S}(\mathfrak{A} \times \mathfrak{B}_2)$ would be in P, for $\mathfrak{B}_2$ is so]. Finally, we prove that $\mathfrak{F} \not\preceq \mathfrak{L} \triangleq ((\mathfrak{B}[(\times\mathfrak{E})]) \times \mathfrak{C})$, by contradiction. For suppose $\mathfrak{F} \preceq \mathfrak{L}$, in which case $\mathfrak{F}$ is in $\mathsf{SKSL} \not\ni \mathfrak{K}_{4:1}$, for $\mathfrak{L}$ is so, and so is $\mathfrak{A} \in \mathbf{H}\mathfrak{F}$. Hence, $\mathfrak{A} \notin \mathbf{I}\mathfrak{K}_{4:1}$. Then, in the []-optional case, $\mathfrak{A}$, being a subalgebra of the two-element algebra $\mathfrak{K}_2$, is not isomorphic to the three-element one $\mathfrak{S}_3$, for $3 \nleqslant 2$, and so $\mathfrak{B} = \mathfrak{K}_1$. Otherwise, $\mathfrak{C} \in \mathsf{KL}$, in which case $\mathfrak{L} \in \mathsf{KL}$, and so $\mathfrak{B} = \mathfrak{K}_1$, for $\mathfrak{A} \in \mathbf{HS}\mathfrak{L} \subseteq \mathsf{KL} \not\ni \mathfrak{S}_3$ is not isomorphic to $\mathfrak{S}_3$. Thus, $\mathfrak{B} = \mathfrak{K}_1$, anyway, in which case, by (2.7), $(\pi_0^{2[(+1)]}[(\times\pi_1^{2[(+1)]})]) \times (\pi_{1[(+1)]}^{2[(+1)]}$ is an isomorphism from $\mathfrak{L}$ onto $\mathfrak{M} \triangleq ([(\mathfrak{K}_3\times)]\mathfrak{C})$, contrary to the assumption that $\mathfrak{F} \not\preceq \mathfrak{M}$ [(for $\mathfrak{C} = \mathfrak{B}_2$)], and so this contradiction shows that $\mathfrak{F} \not\preceq \mathfrak{L}$. Then, by Corollary 3.12 and Remark 4.2, $\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$, as required. □

This, by (2.10) and Footnote 1, does provide a comprehensive hierarchical characterization of relative subdirectly-irreducibles of quasi-varieties studied in the next subsection, for these are generated by subsets of $\mathsf{QMS} \triangleq (\mathsf{AQMS} \cup (\mathsf{AQMS} \otimes \{\mathfrak{B}_2\}) \cup \{\mathfrak{B}_2, \mathfrak{S}_3, \mathfrak{B}_2 \oplus 1, \mathfrak{MS}_{5:1}\})$, where $\mathsf{AQMS} \triangleq (((\mathsf{MS} \cap \mathsf{DML}) \setminus \mathsf{BL}) \cup \{\mathfrak{K}_2, \mathfrak{K}_{4:1}, \mathfrak{K}_3 \oplus 1\})$. To make it clearer, we present several almost immediate consequences of Theorem 5.42, immediately yielding the following two ones, in view of mutually embedable finite algebras' being isomorphic and Lemma 5.41/5.29 as well as absence of proper /isomorphic subalgebras of $\mathfrak{B}_2$/"finite algebras and the inequalities $8 > 4 > 2$":

Corollary 5.43. *Let* $\mathsf{P} \subseteq \mathsf{MSL}$ *be a pre-variety,* $\mathfrak{F} \in \mathbf{S}_{>1}((\mathfrak{S}|\mathfrak{B})_{3|2} \times \mathfrak{B}_2)$ *and* $\mathfrak{K} \triangleq (\pi_0^2{\restriction}F)[\mathfrak{F}] \preceq \| \succeq | = ((\mathfrak{S}\|\mathfrak{B})|\mathfrak{B})_{(3\|2)|2}$. *Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow (\mathfrak{F} \in \mathbf{I}[\mathbf{S}]\mathfrak{K})$.

Corollary 5.44. *Let* $\mathsf{P} \subseteq \mathsf{MSL}$ *be a pre-variety,* $\mathfrak{F} \in (\mathbf{S}_{>1}(\mathfrak{DM}_4 \times \mathfrak{B}_2) \setminus \mathbf{IS}(\mathfrak{K}_3 \times \mathfrak{B}_2)) = \{\mathfrak{DM}_4 \times \mathfrak{B}_2\}$. *Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow (\mathfrak{DM}_4 \notin \mathsf{P})$.

Corollary 5.45. *Let* $\mathsf{P} \subseteq \mathsf{MSL}$ *be a pre-variety,* $\mathfrak{C} \triangleq (\mathfrak{K}|\mathfrak{B})_{3|2}$, $\mathfrak{L} \in (\{\mathfrak{C} \oplus 1\} \cup (\varnothing|\{\mathfrak{K}_2 \times \mathfrak{C}\})$ *and* $\mathfrak{F} \in \mathbf{S}_{>1}\mathfrak{L}$. *Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow ((\mathfrak{K}_2 \in \mathsf{P}) \Rightarrow (\mathfrak{F} \preceq \mathfrak{C}))$. *In particular,* $(\mathfrak{L} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow (\mathfrak{K}_2 \notin \mathsf{P} \ni \mathfrak{L})$.

Proof. First, since $\mathfrak{K} \triangleq (\pi_0{\restriction}F)[\mathfrak{F}] \in \mathbf{S}\mathfrak{K}_2 \subseteq \mathsf{TNIMSL}$, by Corollary 5.4, Lemma 5.15 and (5.9), $\mathfrak{F} \notin \mathsf{TNIMSL} \supseteq \mathbf{IS}\mathfrak{K}$, for $|F| > 1$, is not embedable into $\mathfrak{K}$. Next, if $0 \in K$, then $\mathfrak{K} = \mathfrak{K}_2$. Otherwise, $\mathfrak{K} = \mathfrak{K}_1 \in \mathsf{OMSL} \subseteq \mathsf{P}$, in which case $\pi_1{\restriction}F$ is an embedding of $\mathfrak{F}$ into $\mathfrak{C}$, and so Theorem 5.42 and the fact that $|C| \leqslant 3 < 4 = |L|$ complete the argument. □

Corollary 5.46. *Let* $\mathsf{P} \subseteq \mathsf{MSL}$ *be a pre-variety,* $(\mathfrak{C}|\mathfrak{L}) \triangleq (\mathfrak{B}|\mathfrak{K})_{2|(3[+1:1])}$, $\mathfrak{M} \triangleq (\mathfrak{L} \times \mathfrak{C})$, $\mathsf{K} \triangleq \{\mathfrak{C}[, \mathfrak{K}_3, \mathfrak{S}_3]\}$, $\mathfrak{F} \in ((\mathsf{P} \cap \mathbf{S}_{>1}\mathfrak{M}) \setminus \mathbf{IS}(\mathsf{K} \otimes \{\mathfrak{C}\}))$ *and* $\mathfrak{K} \triangleq (\pi_0^2{\restriction}F)[\mathfrak{F}]$. *Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow ((\mathfrak{F} \notin \mathbf{I}(\{\mathfrak{C}\}[\cap\varnothing])) \Rightarrow ((\mathfrak{L} \notin \mathsf{P}) \& (\mathfrak{F} \in \{\mathfrak{M}, \Re(\mathfrak{L})\})))$.

Proof. The "if" part is by Theorem 5.42, the simplicity of two-element algebras, the inclusions $\mathrm{Si}(\mathsf{P}) \subseteq \mathrm{SI}(\mathsf{P}) \subseteq \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$ and the fact that $(\mathfrak{F} \in \{\mathfrak{M}, \Re(\mathfrak{L})\}) \Rightarrow (\mathfrak{K} = \mathfrak{L})$. Conversely, assume $\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$, in which case $\mathfrak{K} \in (\{\mathfrak{L}\} \cup (\{\mathfrak{K}_1\}[\cap\varnothing]))$, and so $\mathfrak{F} = (\mathfrak{K}_1 \times \mathfrak{C}) \in \mathbf{I}\mathfrak{C}$, unless $\mathfrak{K} = \mathfrak{L}$. Otherwise, $\mathfrak{F} \in \{\mathfrak{M}, \Re(\mathfrak{L})\}$, in which case $\mathfrak{F} \not\preceq \{\mathfrak{K}, \mathfrak{C}\}$, for $|F| \geqslant (4[+1]) > ((3[+1])\|2) = |K\|C|$, and so, by Theorem 5.42, $\mathfrak{L} \notin \mathsf{P}$. □

Corollary 5.47. *Let* $\mathsf{P} \subseteq \mathsf{MSL}$ *be a pre-variety,* $\mathfrak{C} \triangleq \mathfrak{B}_2$, $\mathfrak{I} \triangleq (\mathfrak{K}_3 \oplus 1)$, $\mathfrak{J} \triangleq (\mathfrak{K}_3 \times \mathfrak{C})$, $\mathfrak{L} \triangleq (\mathfrak{K}_2 \times \mathfrak{C})$, $\mathfrak{M} \triangleq (\mathfrak{I} \times \mathfrak{C})$, $\mathfrak{N} \triangleq (\mathfrak{C} \oplus 1)$, $\mathsf{K} \triangleq \{\mathfrak{J}, \mathfrak{N}\}$, $\mathfrak{F} \in (\mathsf{P} \cap (\mathbf{S}_{>1}(\mathfrak{M}) \setminus \mathbf{IS}\mathsf{K}))$, $\mathfrak{G} \triangleq ((\pi_0^3{\restriction}F) \odot (\pi_2^3{\restriction}F))[\mathfrak{F}]$ *and* $\mathfrak{H} \triangleq ((\pi_0^3{\restriction}F) \odot (\pi_1^3{\restriction}F))[\mathfrak{F}] = (\pi_0^2{\restriction}F)[\mathfrak{F}]$ *(cf.* (2.7)*).* *Then,* $(\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})) \Leftrightarrow ((\mathfrak{F} = \mathfrak{M}) \& ((\{\mathfrak{L}, \mathfrak{I}\} \cap \mathsf{P}) = \varnothing))$.

Proof. First, if 0 was not in $\pi_0^3[F]$, then, by (2.1) and (2.7), $(\pi_1^3\restriction F)\times(\pi_2^3\restriction F)$ would be an embedding of $\mathfrak{F}$ into $\mathfrak{J}$. Therefore, $0\in\pi_0^3[F]$, in which case $(\pi_0^3\restriction F)\in\hom^{\mathrm{S}}(\mathfrak{F},\mathfrak{K}_2)$, for $\neg^{\mathfrak{K}_2}0=1$, and so $\mathfrak{F}$ is not in $\mathsf{QSMSL}\ni\mathfrak{C}$, for $\mathfrak{K}_2$ is not so. Hence, $\mathfrak{F}\not\cong\mathfrak{C}$, while, by Lemma 5.36, $\mathfrak{N}\in\mathbf{IS}\mathfrak{F}\subseteq\mathsf{P}$. Next, if there was any $e\in\hom_{\mathrm{I}}(\mathfrak{F},\mathfrak{I})$, then $e[\mathfrak{F}]\in(\mathbf{I}(\mathfrak{F})\cap\mathbf{S}(\mathfrak{I}))\subseteq\mathbf{S}(\mathfrak{K}_2\times\mathfrak{K}_3)$ would be neither idempotent nor quasi-strong, for $\mathfrak{F}\in\mathbf{S}\mathfrak{M}$ is so, in view of Lemma 5.1, in which case we would have $0\in\pi_0[e[F]]$, for $\mathfrak{K}_1\in\mathsf{OMSL}\subseteq\mathsf{QSMSL}\ni\mathfrak{K}_3$, but $1\notin\pi_1[e[F]]$, for, otherwise, we would then get $e[\mathfrak{F}]\not\models(5.1)[x_i/\langle 1-i,1-i\rangle]_{i\in 2}$, and so, by (5.20), $e\circ(\epsilon_2^3\oplus 1)^{-1}$ would be an an embedding of $\mathfrak{F}$ into $\mathfrak{N}$. Likewise, if there was any $f\in\hom_{\mathrm{I}}(\mathfrak{F},\mathfrak{L})$, then $f[\mathfrak{F}]\in(\mathbf{I}(\mathfrak{F})\cap\mathbf{S}(\mathfrak{L}))$ would be a model of (4.13), for $\mathfrak{F}\in\mathbf{S}\mathfrak{M}$ is so, in which case $\langle 0,1\rangle$ would not be in $f[F]$, i.e., $f[\mathfrak{F}]$ would be a subalgebra of $\mathfrak{N}$, for, otherwise, we would have $f[\mathfrak{F}]\not\models(4.13)[x_0/\langle 0,1\rangle]$, and so e would be an an embedding of $\mathfrak{F}$ into $\mathfrak{N}$. Finally, $(\langle 0,0,1\rangle\in F)\Leftrightarrow(\mathfrak{F}=\mathfrak{M})$, for $\mathfrak{M}$ is generated by $\langle 0,0,1\rangle$. Consider the corresponding complementary cases:

- $\mathfrak{F}=\mathfrak{M}$,
  in which case $\mathfrak{G}=\mathfrak{L}$, for $\mathfrak{L}$ is generated by $\langle 0,1\rangle\in G\subseteq L$, while $\mathfrak{H}=\mathfrak{I}$.
- $\mathfrak{F}\neq\mathfrak{M}$,
  in which case $\mathfrak{G}=\mathfrak{N}$, for $\mathfrak{N}$ is generated by $\langle 0,0\rangle\in G\subseteq N$, because $\langle 0,1\rangle\notin G\subseteq L$ but $0\in\pi_0^3[F]=\pi_0^2[G]$.

Then, Theorem 5.42 completes the argument. □

Corollary 5.48. *Let* $\mathsf{P}\subseteq\mathsf{MSL}$ *be a pre-variety,* $\mathfrak{C}\triangleq\mathfrak{S}_3$, $g\triangleq\chi_3^{3\backslash 1}$, $\mathfrak{I}\triangleq(\mathfrak{K}_2\times\mathfrak{B}_2)$, $\mathfrak{L}\triangleq(\mathfrak{C}\oplus 1)$, $\mathfrak{M}\triangleq\mathfrak{M}\mathfrak{S}_{5:1}$, $\mathfrak{N}\triangleq(\mathfrak{B}_2\oplus 1)$, $\mathfrak{F}\in(\mathsf{P}\cap(\mathbf{S}_{>1}(\mathfrak{M})\backslash\mathbf{IS}(\mathfrak{N})))$, $\mathfrak{K}\triangleq(\pi_0^2\restriction F)[\mathfrak{F}]$ *and* $\mathfrak{G}\triangleq((\pi_0^2\restriction F)\odot((\pi_1^2\restriction F)\circ g))[\mathfrak{F}]$ *(cf.* (2.7)*). Then,* $(\mathfrak{F}\in\mathrm{SI}_{\mathsf{P}}(\mathsf{P}))\Leftrightarrow((\mathfrak{G}\in\mathsf{P})\Rightarrow(\mathfrak{F}\preceq\mathfrak{C}))$*, in which case, providing* $\mathfrak{F}=|\neq\mathfrak{M}$, $(\mathfrak{F}\in\mathrm{SI}_{\mathsf{P}}(\mathsf{P}))\Leftrightarrow((\mathfrak{I}\notin\mathsf{P})|(\mathfrak{F}\preceq\mathfrak{C}))$*, and so* $\mathfrak{L}\notin\mathrm{SI}_{\mathsf{P}}(\mathsf{P})$*. In particular,* $(\mathfrak{M}\in\mathrm{SI}_{\mathsf{P}}(\mathsf{P}))\Leftrightarrow(\mathfrak{I}\notin\mathsf{P})$.

Proof. First, if $\mathfrak{F}$ was embedable into $\mathfrak{I}\in\mathsf{ABL}\subseteq\mathsf{NDML}\not\ni\mathfrak{C}$, then $\mathfrak{J}\in(\mathbf{H}(\mathfrak{F})\cap\mathbf{S}(\mathfrak{C}))$ would be distinct from $\mathfrak{C}$, i.e., 1 would not be in J, for $\mathfrak{C}$ is generated by 1, in which case $g\restriction J$ would be injective, and so, by (2.1), (2.7) and (4.21), $(\pi_0^2\restriction F)\odot((\pi_1^2\restriction F)\circ g)$ would be an embedding of $\mathfrak{F}$ into $\mathfrak{N}$, for $\langle 0,1\rangle\notin F\subseteq MS_{5:1}\not\ni\langle 0,2\rangle$. Likewise, if $\mathfrak{F}\in\mathsf{QSMSL}\not\ni\mathfrak{K}_2$, then $\mathsf{QSMSL}\supseteq\mathbf{H}(\mathfrak{F})\ni\mathfrak{K}\in\mathbf{S}(\mathfrak{K}_2)=\{\mathfrak{K}_2,\mathfrak{K}_1\}$, in which case $\mathfrak{K}=\mathfrak{K}_1$, and so π_1^2 is an embedding of $\mathfrak{F}$ into $\mathfrak{C}$. Finally, if $\mathfrak{F}=|\neq\mathfrak{N}$, then $(\mathfrak{G}\in(\{\mathfrak{I}\}|\mathbf{S}\{\mathfrak{N}\}$. Then, Lemmas 5.15, 5.36, (5.8), Theorem 5.42 and the fact that $|M\|L|=(5\|4)>3=|S_3|$ complete the argument. □

5.6. **Lattices of quasi-varieties of quasi-strong Morgan-Stone and almost quasi-strong Kleene-Stone lattices.**

Theorem 5.49. *Pre-//quasi-varieties of /almost [{quasi-}strong]|| Morgan-/Kleene[-Stone]|Stone|Boolean lattices form the/a non-chain /non-distributive lattice*

TABLE 3. Quasi-identities false/true in generating algebras and embedability lemmas for them.

$\mathfrak{K}_2$	(5.7)	$[x_0/0]$	$\varnothing$	5.32
$\mathfrak{K}_{4:1}$	((5.6))	$[x_0/2,x_1/3,x_2/1]$	$\varnothing$	5.30
$\mathfrak{K}_2\times\mathfrak{B}_2$	((4.13))	$[x_0/\langle 0,1\rangle]$	{(5.1)}	5.35
$\mathfrak{S}_3$	(4.19)	$[x_i/(1+i)]_{i\in 2}$	{(4.16)}	5.23
$(\mathfrak{K}\restriction\mathfrak{B})_{3\restriction 2}\oplus 1$	$\lfloor(5.6)\rfloor\restriction(4.11)$	$[x_i/\langle i,i\rangle]_{i\in(2\restriction 1)}$	{(5.7)}	5.36
$\mathfrak{K}_{3\langle+1:1\rangle}\times\mathfrak{B}_2$	$\mathfrak{R}^{\mathrm{W}}_{\langle\mathrm{M}\rangle}$	$[x_i/\langle(4-i) \bmod 3,1\rangle]_{i\in(2\langle+1\rangle)}$	$\{\mathfrak{K}_{\mathrm{M}},((5.6))\}$	5.26
$\mathfrak{K}_{5:1}$	(4.12)	$[x_0/3,x_1/1]$	$\{(4.16),(\neg x_2\approx x_2)\to\mathfrak{R}^{\mathrm{W}}\}$	5.31
$(\mathfrak{K}\restriction\mathfrak{B})_{3\restriction 2}$	$(5.4)\restriction(4.14)$	$[x_i/(1-i)]_{i\in(2\restriction 1)}$	$\varnothing$	5.3
$(\mathfrak{K}_3\oplus 1)\times\mathfrak{B}_2$	$\mathfrak{R}^{\mathrm{QS}}_{\mathrm{N}}$	$[x_i/\langle 1-i,1-i,1\rangle]_{i\in 2}$	{(5.1),((4.13))}	5.39
$\mathfrak{M}\mathfrak{S}_{5:1}$	(4.13)	$[x_0/\langle 0,1\rangle]$	{((4.13))}	5.38
$\mathfrak{K}_4$	$\lceil(4.20)\rceil$	$[x_i/(2+i)]_{i\in 2}$	{(5.1)}	5.27
$\mathfrak{D}\mathfrak{M}_4$	(5.5)	$[x_i/\langle i,1-i\rangle]_{i\in 2}$	{(4.16)}	5.28
$\mathfrak{D}\mathfrak{M}_4\times\mathfrak{B}_2$	$\mathfrak{K}$	$[x_i/\langle i,1-i,1\rangle]_{i\in 2}$	{(4.16),(5.1)}	5.29

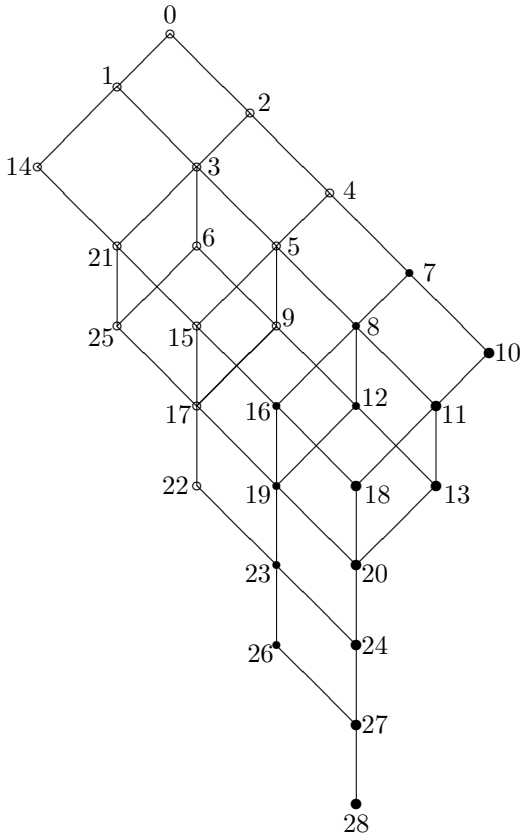

FIGURE 4. The lattice of quasi-varieties of quasi-strong Morgan-Stone lattices.

with $((8[+(\{3\cdot\}7)])|3|2)/(([\{50+\}36+]21)|11|6)$ *elements "as well as Hasse diagram with [either both small and* $\{$*either non-solid or*$\}$ *solid or] both solid and large circles-nodes depicted at Figure 4, in which case it is embedable into* $[\mathfrak{D}_{2\{+3\}}\times](\mathfrak{D}_5\times\mathfrak{D}_3)$*, and so is distributive"*$||/$ *[*$\{$*the list of which, numbered from* $0/14$ *to* $28/120$*, is given by the first and second columns of Tables 4 and 5, in which case, for each* $i\in 121$*,* $f(i)$ *is generated by* $\pi_1[g(i)]$*, where* $f\|g$ *is the function with domain* 123 *given by the first and second*$\|$*third columns of Tables 4 and 5, and so* $\mathrm{SI}_{f(i)}(f(i)) = (\mathbf{I}(((\mathsf{MS}\cup\mathsf{K}')\cap\mathbf{IS}\pi_1[g(i)])\cup(\mathbf{S}_{>1}(\pi_1[g(i)]\setminus\mathsf{K}')\setminus\mathbf{IS}(\{\mathfrak{S}_3,\mathfrak{B}_2\}\otimes\{\mathfrak{B}_2\}))))\setminus\mathbf{I}(\bigcup\{\mathsf{K}\mid(\{\mathfrak{K}_3,\mathfrak{DM}_4,\mathfrak{K}_3\oplus 1,\mathfrak{K}_{4:1}\}\cap\pi_1[g(i)])\neq\varnothing\})$*, where* $\mathsf{K}\triangleq\{\mathfrak{K}_3\times\mathfrak{B}_2,\mathfrak{K}_4\}$ *and* $\mathsf{K}'\triangleq(\mathsf{K}\cup\{\mathfrak{K}_2\times\mathfrak{B}_2,\mathfrak{B}_2\oplus 1,\mathfrak{K}_3\oplus 1,(\mathfrak{K}_3\oplus 1)\times\mathfrak{B}_2,\mathfrak{MS}_{5:1}\})$*, while, for every* $\langle j,\mathfrak{A}\rangle\in g(i)$*, both* $f(j)=(f(i)\cap\mathrm{Mod}(e(\mathfrak{A})))$ *and* $(f(i)\setminus f(j))\subseteq\mathrm{Mod}(t(\mathfrak{A}))$ *but* $\mathfrak{A}\not\models e(\mathfrak{A})[h(\mathfrak{A})]$*, where* $e\wr h\wr t$ *is the function with domain* QMS *given by the first and second*$\wr$*third*$\wr$*fourth columns of Table 3, in which case* $\mathfrak{A}\notin f(j)\subseteq f(i)\ni\mathfrak{A}$*, and so* $f(j)\subsetneq f(i)$*. In particular,* $f\restriction(29/(121\setminus 14))$ *is an isomorphism from the poset over* $29/(121\setminus 14)$ *with partial ordering, being the union of* $\Delta_{29/(121\setminus 14)}$ *and the transitive closure of* $(\varnothing/\{\langle 112,29\rangle\})\cup(\bigcup\{\pi_0[g(k)]\times\{k\}\mid k\in(29/(121\setminus 14))\}$*, onto the lattice poset involved.*$\}$*]*$||$

Proof. We use Lemma 4.6 and Corollary 4.7 tacitly. Consider any $i\in 121$ and any $\langle j,\mathfrak{A}\rangle\in g(i)$, in which case the falsity of $e(\mathfrak{A})$ in $\mathfrak{A}$ under $h(\mathfrak{A})$ is immediate, while the generation of $f(i)$ by $\pi_1[g(i)]$ is by Corollaries 5.4, 5.5, 5.6, 5.8, 5.14, 5.19, 5.20, 5.21, 5.18, Theorems 5.12, 5.17, 5.24, 5.25 and Lemma 5.7, whereas

TABLE 4. Quasi-varieties of /almost quasi-strong Morgan-/Kleene-Stone lattices and their generating sets (part I: 0–79).

0	QSMSL	$\{\langle 1, \mathbb{D}\mathfrak{M}_4\rangle, \langle 2, \mathfrak{K}_{4:1}\rangle\}$
1	NIQSMSL ∪ QSKSL	$\{\langle 14, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 3, \mathfrak{K}_{4:1}\rangle\}$
2	NIQSMSL ∪ DML	$\{\langle 3, \mathbb{D}\mathfrak{M}_4\rangle, \langle 4, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
3	NIQSMSL ∪ KL	$\{\langle 21, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 5, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 6, \mathfrak{K}_3\rangle\}$
4	MRQSMSL	$\{\langle 5, \mathbb{D}\mathfrak{M}_4\rangle, \langle 7, \mathfrak{K}_{5:1}\rangle\}$
5	NIMRQSMSL ∪ KL	$\{\langle 15, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 8, \mathfrak{K}_{5:1}\rangle, \langle 9, \mathfrak{K}_3\rangle\}$
6	NIQSMSL	$\{\langle 9, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 25, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle\}$
7	SMSL	$\{\langle 8, \mathbb{D}\mathfrak{M}_4\rangle, \langle 10, \mathfrak{S}_3\rangle\}$
8	NISMSL ∪ SKSL	$\{\langle 16, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 12, \mathfrak{K}_3\rangle, \langle 11, \mathfrak{S}_3\rangle\}$
9	NIMRQSMSL	$\{\langle 17, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 12, \mathfrak{K}_{5:1}\rangle\}$
10	DML	$\{\langle 11, \mathbb{D}\mathfrak{M}_4\rangle\}$
11	NIDML ∪ KL	$\{\langle 18, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 13, \mathfrak{K}_3\rangle\}$
12	NISMSL	$\{\langle 19, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle, \langle 13, \mathfrak{S}_3\rangle\}$
13	NIDML	$\{\langle 20, \mathbb{D}\mathfrak{M}_4 \times \mathfrak{B}_2\rangle\}$
14	QSKSL	$\{\langle 21, \mathfrak{K}_{4:1}\rangle\}$
15	MRQSKSL	$\{\langle 17, \mathfrak{K}_3\rangle, \langle 16, \mathfrak{K}_{5:1}\rangle\}$
16	SKSL	$\{\langle 18, \mathfrak{S}_3\rangle, \langle 19, \mathfrak{K}_3\rangle\}$
17	NIMRQSKSL	$\{\langle 22, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle, \langle 19, \mathfrak{K}_{5:1}\rangle\}$
18	KL	$\{\langle 20, \mathfrak{K}_3\rangle\}$
19	NISKSL	$\{\langle 20, \mathfrak{S}_3\rangle, \langle 23, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
20	NIKL	$\{\langle 24, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
21	NIQSKSL ∪ KL	$\{\langle 15, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 25, \mathfrak{K}_3\rangle\}$
22	RQSKSL	$\{\langle 23, \mathfrak{K}_{5:1}\rangle\}$
23	RSKSL	$\{\langle 24, \mathfrak{S}_3\rangle, \langle 26, \mathfrak{K}_4\rangle\}$
24	RKL	$\{\langle 27, \mathfrak{K}_4\rangle\}$
25	NIQSKSL	$\{\langle 17, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
26	SL	$\{\langle 27, \mathfrak{S}_3\rangle\}$
27	BL	$\{\langle 28, \mathfrak{B}_2\rangle\}$
28	OMSL	$\varnothing$
29	AQSKSL	$\{\langle 30, \mathfrak{K}_2\rangle, \langle 31, \mathfrak{K}_{4:1}\rangle\}$
30	NIAQSKSL ∪ VQSKSL	$\{\langle 32, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 33, \mathfrak{K}_{4:1}\rangle, \langle 34, \mathfrak{K}_3 \oplus 1\rangle\}$
31	NIAQSKSL ∪ AKL	$\{\langle 33, \mathfrak{K}_2\rangle, \langle 35, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 36, \mathfrak{K}_3\rangle\}$
32	PVQSKSL	$\{\langle 37, \mathfrak{MS}_{5:1}\rangle, \langle 38, \mathfrak{K}_{4:1}\rangle, \langle 39, \mathfrak{K}_3 \oplus 1\rangle\}$
33	NIAQSKSL ∪ VKL	$\{\langle 38, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 40, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 41, \mathfrak{K}_3 \oplus 1\rangle\}$
34	NIAQSKSL ∪ QSKSL	$\{\langle 39, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 41, \mathfrak{K}_{4:1}\rangle\}$
35	NIWMRAQSKSL ∪ AKL	$\{\langle 40, \mathfrak{K}_2\rangle, \langle 42, \mathfrak{K}_{5:1}\rangle, \langle 43, \mathfrak{K}_3\rangle\}$
36	NIAQSKSL ∪ TNIMSL	$\{\langle 44, \mathfrak{K}_2\rangle, \langle 43, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
37	VQSKSL	$\{\langle 45, \mathfrak{K}_{4:1}\rangle, \langle 46, \mathfrak{K}_3 \oplus 1\rangle\}$
38	NIPVQSKSL ∪ VKL	$\{\langle 45, \mathfrak{MS}_{5:1}\rangle, \langle 47, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 48, \mathfrak{K}_3 \oplus 1\rangle\}$
39	NIPVQSKSL ∪ QSKSL	$\{\langle 46, \mathfrak{MS}_{5:1}\rangle, \langle 48, \mathfrak{K}_{4:1}\rangle, \langle 49, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
40	NIWMRAQSKSL ∪ VKL	$\{\langle 47, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 50, \mathfrak{K}_{5:1}\rangle, \langle 51, \mathfrak{K}_3 \oplus 1\rangle\}$
41	NIAQSKSL ∪ KL	$\{\langle 48, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 51, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 44, \mathfrak{K}_3\rangle\}$
42	NIASKSL ∪ AKL	$\{\langle 50, \mathfrak{K}_2\rangle, \langle 52, \mathfrak{S}_3\rangle, \langle 53, \mathfrak{K}_3\rangle\}$
43	NIWMRAQSKSL ∪ TNIMSL	$\{\langle 54, \mathfrak{K}_2\rangle, \langle 53, \mathfrak{K}_{5:1}\rangle, \langle 55, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
44	NIAQSKSL	$\{\langle 56, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 54, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
45	NIVQSKSL ∪ VKL	$\{\langle 58, \mathfrak{K}_3 \oplus 1\rangle, \langle 57, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
46	NIVQSKSL ∪ QSKSL	$\{\langle 59, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 58, \mathfrak{K}_{4:1}\rangle\}$
47	NIWMRPVQSKSL ∪ VKL	$\{\langle 57, \mathfrak{MS}_{5:1}\rangle, \langle 60, \mathfrak{K}_{5:1}\rangle, \langle 61, \mathfrak{K}_3 \oplus 1\rangle\}$
48	NIPVQSKSL ∪ KL	$\{\langle 58, \mathfrak{MS}_{5:1}\rangle, \langle 61, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 62, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 56, \mathfrak{K}_3\rangle\}$
49	NINQSRPVQSKSL ∪ QSKSL	$\{\langle 59, \mathfrak{MS}_{5:1}\rangle, \langle 62, \mathfrak{K}_{4:1}\rangle\}$
50	NIASKSL ∪ VKL	$\{\langle 60, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 63, \mathfrak{S}_3\rangle, \langle 64, \mathfrak{K}_3 \oplus 1\rangle\}$
51	NIWMRAQSKSL ∪ KL	$\{\langle 61, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 64, \mathfrak{K}_{5:1}\rangle, \langle 54, \mathfrak{K}_3\rangle\}$
52	AKL	$\{\langle 63, \mathfrak{K}_2\rangle, \langle 65, \mathfrak{K}_3\rangle\}$
53	NIASKSL ∪ TNIMSL	$\{\langle 66, \mathfrak{K}_2\rangle, \langle 65, \mathfrak{S}_3\rangle, \langle 67, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
54	NIWMRAQSKSL	$\{\langle 68, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 66, \mathfrak{K}_{5:1}\rangle, \langle 69, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
55	WRAQSKSL	$\{\langle 69, \mathfrak{K}_2\rangle, \langle 67, \mathfrak{K}_{5:1}\rangle\}$
56	NIPVQSKSL	$\{\langle 70, \mathfrak{MS}_{5:1}\rangle, \langle 68, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 71, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
57	NIWMRVQSKSL ∪ VKL	$\{\langle 72, \mathfrak{K}_{5:1}\rangle, \langle 73, \mathfrak{K}_3 \oplus 1\rangle\}$
58	NIVQSKSL ∪ KL	$\{\langle 73, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 74, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 70, \mathfrak{K}_3\rangle\}$
59	NIQSRVQSKSL ∪ QSKSL	$\{\langle 74, \mathfrak{K}_{4:1}\rangle, \langle 14, \mathfrak{B}_2 \oplus 1\rangle\}$
60	NIPVSKSL ∪ VKL	$\{\langle 72, \mathfrak{MS}_{5:1}\rangle, \langle 75, \mathfrak{K}_3 \oplus 1\rangle\}$
61	NIWMRPVQSKSL ∪ KL	$\{\langle 73, \mathfrak{MS}_{5:1}\rangle, \langle 75, \mathfrak{K}_{5:1}\rangle, \langle 76, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 68, \mathfrak{K}_3\rangle\}$
62	NINQSRPVQSKSL ∪ KL	$\{\langle 74, \mathfrak{MS}_{5:1}\rangle, \langle 76, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 71, \mathfrak{K}_3\rangle\}$
63	NIAKL ∪ VKL	$\{\langle 77, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 78, \mathfrak{K}_3 \oplus 1\rangle\}$
64	NIASKSL ∪ KL	$\{\langle 75, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 78, \mathfrak{S}_3\rangle, \langle 66, \mathfrak{K}_3\rangle\}$
65	NIAKL ∪ TNIMSL	$\{\langle 79, \mathfrak{K}_2\rangle, \langle 80, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
66	NIASKSL	$\{\langle 81, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 79, \mathfrak{S}_3\rangle, \langle 82, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
67	WRASKSL	$\{\langle 82, \mathfrak{K}_2\rangle, \langle 80, \mathfrak{S}_3\rangle, \langle 83, \mathfrak{K}_4\rangle\}$
68	NIWMRPVQSKSL	$\{\langle 84, \mathfrak{MS}_{5:1}\rangle, \langle 81, \mathfrak{K}_{5:1}\rangle, \langle 85, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
69	NIWRAQSKSL	$\{\langle 99, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 82, \mathfrak{K}_{5:1}\rangle\}$
70	NIVQSKSL	$\{\langle 88, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 84, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
71	NINQSRPVQSKSL	$\{\langle 88, \mathfrak{MS}_{5:1}\rangle, \langle 85, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
72	NIVSKSL ∪ VKL	$\{\langle 77, \mathfrak{S}_3\rangle, \langle 89, \mathfrak{K}_3 \oplus 1\rangle\}$
73	NIWMRVQSKSL ∪ KL	$\{\langle 89, \mathfrak{K}_{5:1}\rangle, \langle 90, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 84, \mathfrak{K}_3\rangle\}$
74	NIQSRVQSKSL ∪ KL	$\{\langle 21, \mathfrak{B}_2 \oplus 1\rangle, \langle 90, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle, \langle 88, \mathfrak{K}_3\rangle\}$
75	NIPVSKSL ∪ KL	$\{\langle 89, \mathfrak{MS}_{5:1}\rangle, \langle 81, \mathfrak{K}_3\rangle, \langle 91, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
76	NINQSRWMRPVQSKSL ∪ KL	$\{\langle 90, \mathfrak{MS}_{5:1}\rangle, \langle 91, \mathfrak{K}_{5:1}\rangle, \langle 85, \mathfrak{K}_3\rangle\}$
77	VKL	$\{\langle 93, \mathfrak{K}_3 \oplus 1\rangle\}$
78	NIAKL ∪ KL	$\{\langle 93, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 79, \mathfrak{K}_3\rangle\}$
79	NIAKL	$\{\langle 94, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 86, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$

the fact that both $f(j) = (f(i) \cap \mathrm{Mod}(e(\mathfrak{A})))$ and $(f(i) \setminus f(j)) \subseteq \mathrm{Mod}(t(\mathfrak{A}))$ is due to Lemmas 5.7, 5.15, 5.22, Theorems 5.24, 5.25, Corollaries 5.5, 5.6, 5.8, 5.11, (4.18), (5.8), (5.9), (5.16), (5.19) and (5.18). Then, for any pre-variety $\mathsf{P} \subseteq f(i)$, by Corollaries 5.5, 5.6, 5.8 as well as Lemmas 5.7, 5.15 and $\boldsymbol{\ell}(\mathfrak{A})$, where the function $\boldsymbol{\ell}$ with domain QMS is given by the first and fifth columns of Table 3, $\mathfrak{A} \in \mathsf{P}$, unless $\mathsf{P} \subseteq f(j)$. In particular, $\mathsf{P} = f(i)$, whenever $\mathsf{P} \subseteq f(k)$, for no $k \in \pi_0[g(i)]$, in which case, $\mathrm{img}\, f$, containing all varieties of almost/ quasi-strong Morgan-/Kleene-Stone lattices, does exhaust all their pre-varieties, because these are subsumed by the varieties they generate, and so Corollaries 5.16, 5.43, 5.44, 5.45, 5.46, 5.47, 5.48, Footnote 1, (2.10), Theorem 4.4 and Table 3 complete the argument. □

This subsumes [18, Theorem 4.8], in its turn, subsuming [1, Theorem 6.2], as well as, by Corollaries 4.7, 5.4, 5.6 and Theorem 5.12, immediately yields:

Corollary 5.50. *Any [pre-/-quasi-]variety* $\mathsf{P} \subseteq \mathsf{SMSL}$ *such that* $\mathsf{P} \nsubseteq \mathsf{DML}$ *is generated by* $(\mathsf{P} \cap \mathsf{DML}) \cup \mathsf{SL}$.

Though we have refrained from explicit presenting the Hasse diagram of the lattice of quasi-varieties of almost quasi-strong Kleene-Stone lattices just because of its being too expansive and complicated, Theorem 5.49 does provide its comprehensive description. The reason of restricting our consideration by merely almost quasi-strong *Kleene*-Stone lattices is clarified by the next subsection.

TABLE 5. Quasi-varieties of /almost quasi-strong Morgan-/Kleene-Stone lattices and their generating sets (part II: 80–120).

80	WRAKL	$\{\langle 86, \mathfrak{K}_2\rangle, \langle 97, \mathfrak{K}_4\rangle\}$
81	NIPVSKSL	$\{\langle 95, \mathfrak{MS}_{5:1}\rangle, \langle 92, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
82	NIWRASKSL	$\{\langle 96, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 86, \mathfrak{S}_3\rangle, \langle 87, \mathfrak{K}_4\rangle\}$
83	ASL	$\{\langle 87, \mathfrak{K}_2\rangle, \langle 97, \mathfrak{S}_3\rangle\}$
84	NIWMRVQSKSL	$\{\langle 95, \mathfrak{K}_{5:1}\rangle, \langle 98, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
85	NINQSRWMRPVQSKSL	$\{\langle 98, \mathfrak{MS}_{5:1}\rangle, \langle 92, \mathfrak{K}_{5:1}\rangle, \langle 99, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
86	NIWRAKL	$\{\langle 100, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 101, \mathfrak{K}_4\rangle\}$
87	NIASL	$\{\langle 102, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 101, \mathfrak{S}_3\rangle\}$
88	NIQSRVQSKSL	$\{\langle 25, \mathfrak{B}_2 \oplus 1\rangle, \langle 98, \mathfrak{K}_{4:1} \times \mathfrak{B}_2\rangle\}$
89	NIVSKSL ∪ KL	$\{\langle 93, \mathfrak{S}_3\rangle, \langle 95, \mathfrak{K}_3\rangle, \langle 103, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
90	NIQSRWMRVQSKSL ∪ KL	$\{\langle 15, \mathfrak{B}_2 \oplus 1\rangle, \langle 103, \mathfrak{K}_{5:1}\rangle, \langle 98, \mathfrak{K}_3\rangle\}$
91	NINQSRPVSKSL ∪ KL	$\{\langle 103, \mathfrak{MS}_{5:1}\rangle, \langle 92, \mathfrak{K}_3\rangle\}$
92	NINQSRPVSKSL	$\{\langle 106, \mathfrak{MS}_{5:1}\rangle, \langle 96, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle\}$
93	NIVKL ∪ KL	$\{\langle 94, \mathfrak{K}_3\rangle, \langle 104, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
94	NIVKL	$\{\langle 105, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
95	NIVSKSL	$\{\langle 94, \mathfrak{S}_3\rangle, \langle 106, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle\}$
96	WRPVSKSL	$\{\langle 107, \mathfrak{MS}_{5:1}\rangle, \langle 102, \mathfrak{K}_4\rangle\}$
97	ABL	$\{\langle 101, \mathfrak{K}_2\rangle, \langle 108, \mathfrak{B}_2\rangle\}$
98	NIQSRWMRVQSKSL	$\{\langle 109, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle, \langle 106, \mathfrak{K}_{5:1}\rangle, \langle 17, \mathfrak{B}_2 \oplus 1\rangle\}$
99	WRPVQSKSL	$\{\langle 109, \mathfrak{MS}_{5:1}\rangle, \langle 96, \mathfrak{K}_{5:1}\rangle\}$
100	NIWRVKL	$\{\langle 110, \mathfrak{K}_4\rangle, \langle 24, \mathfrak{B}_2 \oplus 1\rangle\}$
101	NIABL	$\{\langle 110, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle\}$
102	PVSL	$\{\langle 111, \mathfrak{MS}_{5:1}\rangle\}$
103	NIQSRVSKSL ∪ KL	$\{\langle 106, \mathfrak{K}_3\rangle, \langle 104, \mathfrak{S}_3\rangle, \langle 16, \mathfrak{B}_2 \oplus 1\rangle\}$
104	NIQSRVKL ∪ KL	$\{\langle 105, \mathfrak{K}_3\rangle, \langle 18, \mathfrak{B}_2 \oplus 1\rangle\}$
105	NIQSRVKL	$\{\langle 100, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle, \langle 20, \mathfrak{B}_2 \oplus 1\rangle\}$
106	NIQSRVSKSL	$\{\langle 107, \mathfrak{K}_3 \times \mathfrak{B}_2\rangle, \langle 105, \mathfrak{S}_3\rangle, \langle 19, \mathfrak{B}_2 \oplus 1\rangle\}$
107	WRVSKSL	$\{\langle 111, \mathfrak{K}_4\rangle, \langle 100, \mathfrak{S}_3\rangle, \langle 23, \mathfrak{B}_2 \oplus 1\rangle\}$
108	TNIMSL	$\{\langle 28, \mathfrak{K}_2\rangle\}$
109	WRVQSKSL	$\{\langle 107, \mathfrak{K}_{5:1}\rangle, \langle 22, \mathfrak{B}_2 \oplus 1\rangle\}$
110	VBL	$\{\langle 27, \mathfrak{B}_2 \oplus 1\rangle\}$
111	VSL	$\{\langle 110, \mathfrak{S}_3\rangle, \langle 26, \mathfrak{B}_2 \oplus 1\rangle\}$
112	ASKSL	$\{\langle 53, \mathfrak{K}_3\rangle, \langle 52, \mathfrak{S}_3\rangle, \langle 113, \mathfrak{K}_2\rangle\}$
113	NIASKSL ∪ VSKSL	$\{\langle 114, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 63, \mathfrak{S}_3\rangle, \langle 115, \mathfrak{K}_3 \oplus 1\rangle\}$
114	PVSKSL	$\{\langle 116, \mathfrak{MS}_{5:1}\rangle, \langle 117, \mathfrak{K}_3 \oplus 1\rangle\}$
115	NIASKSL ∪ SKSL	$\{\langle 117, \mathfrak{K}_2 \times \mathfrak{B}_2\rangle, \langle 78, \mathfrak{S}_3\rangle, \langle 66, \mathfrak{K}_3\rangle\}$
116	VSKSL	$\{\langle 77, \mathfrak{S}_3\rangle, \langle 118, \mathfrak{K}_3 \oplus 1\rangle\}$
117	NIPVSKSL ∪ SKSL	$\{\langle 119, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 118, \mathfrak{MS}_{5:1}\rangle, \langle 81, \mathfrak{K}_3\rangle\}$
118	NIVSKSL ∪ SKSL	$\{\langle 120, (\mathfrak{K}_3 \oplus 1) \times \mathfrak{B}_2\rangle, \langle 93, \mathfrak{S}_3\rangle, \langle 95, \mathfrak{K}_3\rangle\}$
119	NINQSRPVSKSL ∪ SKSL	$\{\langle 120, \mathfrak{MS}_{5:1}\rangle, \langle 92, \mathfrak{K}_3\rangle\}$
120	NIQSRVSKSL ∪ SKSL	$\{\langle 104, \mathfrak{S}_3\rangle, \langle 106, \mathfrak{K}_3\rangle, \langle 16, \mathfrak{B}_2 \oplus 1\rangle\}$

5.7. **Infiniteness of the lattice of quasi-varieties of almost De Morgan lattices.** To avoid cumbersome couple notations, we use standard abbreviations: $(\mathsf{f}|\mathsf{t}) \triangleq \langle 0|1, 0|1\rangle$ and $(\mathsf{n}|\mathsf{b}) \triangleq \langle 0|1, 1|0\rangle$.

Given any $n \in (\omega \setminus 1)$, let $\mathcal{Q}_n \triangleq ((\{\neg x_i \approx x_i \mid i \in n\} \cup \{\neg x_n \approx (\vee_+\langle x_j\rangle_{j\in n})\}) \to (\neg\neg x_n \approx x_n))$, in which case $\mathrm{Mod}(\mathcal{Q}_1) = \mathrm{Mod}((5.7))$, and so, by Lemma 5.15,

$$(\mathsf{AV} \cap \mathrm{Mod}(\mathcal{Q}_1)) = (\mathsf{NIAV} \cup \mathsf{VV}), \tag{5.25}$$

where $\mathsf{V} \triangleq (([\{\mathsf{Q}\}\mathsf{S}](\mathsf{M}|\mathsf{K})[\mathsf{S}])/(\mathsf{S}\|\mathsf{B}))\mathsf{L}$, while, for any $\mathsf{K} \subseteq \mathsf{MSL}$, $(\mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}_n)) = (\mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}_{n+1}[x_{n+k}/x_{(n-1)+k}]_{k\in 2}))$, in which case:

$$(\mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}_{n+1})) \subseteq (\mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}_n)), \tag{5.26}$$

and so $\langle \mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}_l)\rangle_{l\in(\omega\setminus 1)}$ is a decreasing denumerable chain of relative sub-quasi-varieties of K with intersection, being that relatively axiomatized by $\boldsymbol{\mathcal{Q}}_\omega \triangleq \{\mathcal{Q}_n \mid n \in (\omega \setminus 1)\}$.

Lemma 5.51. *Let $n \in (\omega \setminus 2)$, $\bar{a} \triangleq \langle\{\langle i, \mathsf{b}\rangle\} \cup ((n \setminus \{i\}) \times \{\mathsf{n}\})\rangle_{i\in n}$ and $\mathfrak{DM}_{4:n}$ the subalgebra of $\mathfrak{DM}_4^n$ generated by $I_n \triangleq (\mathrm{img}\,\bar{a})$. Then, $\mathcal{Q}_{(n-1)[+1]}$ is [not] true in $(\mathfrak{DM}_{4:n} \oplus 1) \in \mathsf{IVDML}$ [under $[x_j/\langle 1, a_j\rangle; x_n/\langle 0, n \times \{\mathsf{f}\}\rangle]_{j\in n}$].*

Proof. Clearly, $A \triangleq DM_{4:n} \ni b \triangleq (\vee_+^{\mathfrak{DM}_4^n}\bar{a}) = (n \times \{\mathsf{t}\}) = \neg^{\mathfrak{DM}_4^n}\neg^{\mathfrak{DM}_4^n} b$, in which case $A \ni c \triangleq \neg^{\mathfrak{DM}_4^n} b = (n \times \{\mathsf{f}\})$, and so the []-optional part holds, for $I_n \subseteq \Im^{\mathfrak{DM}_4^n}$, while $\neg^{\mathfrak{K}_2} 0 = 1 = (\vee_+^{\mathfrak{DM}_4^n}(n \times \{1\})) \in \Im^{\mathfrak{K}_2}$. Now, given any $B \subseteq A$, let $(C|D)(B) \triangleq (\wedge|\vee)_+^{\mathfrak{DM}_4^n}[B^+] \in \wp(B, A)$, in which case $E \triangleq (C(I_n)\setminus I_n) \subseteq \{\mathsf{n}, \mathsf{f}\}^n$, and so $D(E) \subseteq \{\mathsf{n}, \mathsf{f}\}^n$, while, for each $k \in n$ and every $\bar{d} \in E$, either of $d_k = (\mathsf{n}\|\mathsf{f})$ holds, implying, resp., $(a_k \vee^{\mathfrak{DM}_4^n} \bar{d}) \ni \| = (\langle k, \mathsf{t}\rangle|a_k)$ |"because $a_k[n \setminus \{k\}] = \{\mathsf{n}\}$", whereas, for all $l \in (n \setminus \{k\})$, $\pi_{k/l}(a_k \vee^{\mathfrak{DM}_4^n} a_l) = \mathsf{t}$. Then, by (4.1) and (4.5), $A = D(C(I_n))$, in which case, since, by (4.3), the Σ_+^--quasi-identity $(\{\neg x_m \approx x_m \mid m \in 2\} \cup \{x_0 \lessapprox x_1\}) \to (x_1 \approx x_0)$ is true in $\mathsf{MSL} \ni \mathfrak{DM}_4^n$, $(\Im^{\mathfrak{DM}_{4:n}} \setminus I_n) \subseteq D(E)$, and so $I_n \subseteq J \triangleq \Im^{\mathfrak{DM}_{4:n}} \subseteq (I_n \cup \{n \times \{\mathsf{n}\}\})$. Finally, we prove the truth of $\mathcal{Q}_{n-1}$ in $\mathfrak{F} \triangleq (\mathfrak{DM}_{4:n} \oplus 1)$ by contradiction. For suppose there are some $\bar{e} \in (\Im^{\mathfrak{F}})^{n-1}$ and $f \in F$ such that $\neg^{\mathfrak{F}} f = (\vee_+^{\mathfrak{F}}\bar{e})$, in which case $\bar{g} \triangleq (\bar{e} \circ \pi_1) \in J^{n-1}$, while $f = \langle 0, c\rangle$, whereas $(\vee_+^{\mathfrak{DM}_{4:n}}\bar{g}) = b$, and so, for each $\ell \in n$, there is some $\Bbbk_\ell \in (n-1)$ such that $\pi_\ell(b_{\Bbbk_\ell}) = \mathsf{b}$. Then, $b_{\Bbbk_\ell} = a_\ell$, in which case $\bar{\Bbbk} \in (n-1)^n$ is injective, and so $n \leqslant (n-1)$. This contradiction completes the argument. □

This, by Corollary 4.7, Lemma 5.32, (4.18), (5.25), (5.26) and the Compactness Theorem for ultra-multilicative classes (cf., e.g., [15]), immediately yields:

Theorem 5.52. *$\langle \mathsf{ADML} \cap \mathrm{Mod}(\mathcal{Q}_n)\rangle_{n\in(\omega\setminus 1)}$ is a strictly decreasing countable chain of proper sub-quasi-varieties of $\mathsf{ADML} = (\mathsf{AQSMSL} \cap \mathsf{NDML}) \subseteq [\mathsf{PS}]\mathsf{MSL}$, in which case $(\mathsf{ADML} \cap \mathrm{Mod}(\boldsymbol{\mathcal{Q}}_\omega)) = (\mathsf{ADML} \cap \mathrm{Mod}(\mathcal{Q}_m))$, for no $m \in (\omega \setminus 1)$, and so this is not finitely-axiomatizable.*

6. Relatively semi-simple quasi-varieties of Morgan-Stone lattices and algebras

First, by (2.4) as well as Lemmas 4.6, due to which, for any $i \in 2$, $\mathfrak{K}_{3[,01]} \preceq \mathfrak{K}_{4:i[,01]}$, and 5.40, we immediately get:

Corollary 6.1. *Let $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{MSL}$ be a pre-variety and $i \in 2$. Suppose $\mathfrak{K}_{4:i[,01]} \in \mathsf{P}$. Then, $\mathfrak{K}_{4:i[,01]} \in (\mathrm{SI}_\mathsf{P}(\mathsf{P}) \setminus \mathrm{Si}_\mathsf{P}(\mathsf{P}))$.*

Lemma 6.2. *Let $\mathsf{P} \subseteq \mathsf{A}((\mathsf{DM})|\mathsf{K})\mathsf{L}$ be a pre-variety containing $(\mathfrak{DM}|\mathfrak{K})_{4|3}$. Then, $\mathfrak{K}_2$ is embeddable into any $\mathfrak{A} \in \mathrm{Si}_\mathsf{P}(\mathsf{P} \setminus \mathsf{DML})$.*

Proof. Take any $a \in (A \setminus \{\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a\})$. Let $(\mathcal{H}\|\mathcal{G}) \triangleq \hom(\mathfrak{A}, (\mathfrak{K}\|(\mathfrak{DM}|\mathfrak{K}))_{2\|(4|3)})$ and $(\theta\|\vartheta) \triangleq (A^2 \cap (\bigcap \ker[\mathcal{H}\|\mathcal{G}]))$, in which case, by (2.8) and Corollary 4.7, $(\theta \cap \vartheta) = \Delta_A$, while, by (2.5) and the P-simplicity of $\mathfrak{A}$, $\vartheta \in \mathrm{Co}_{\mathsf{P}}(\mathfrak{A}) = \{A^2, \Delta_A\}$, whereas, by Corollary 4.7, $\Delta_A \not\ni \langle a, \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a\rangle \in \vartheta$, and so $\vartheta = A^2$. Then, $\theta = \Delta_A$, in which case, by (2.8), Corollary 4.7 and the P-simplicity of $\mathfrak{A}$, $\mathfrak{A} \in (\mathsf{TNIMSL} \setminus \mathsf{OMSL})$, and so Lemma 5.33 completes the argument. □

The P-simplicity of $\mathfrak{A}$ cannot be neither omitted nor replaced by its idempotence here, even if P is quasi-equational, when taking $\mathfrak{K}_2 \notin \mathsf{P} \triangleq (\mathsf{A}((\mathsf{DM})|\mathsf{K})\mathsf{L} \cap \mathrm{Mod}(\mathfrak{Q}_1)) \ni [\mathfrak{A} \triangleq](((\mathfrak{DM})|\mathfrak{K})_{4|3}[\oplus 1])[\in (\mathsf{IP} \setminus \mathsf{DML})$, for $((\mathfrak{DM})|\mathfrak{K})_{4|3}$ is idempotent, while $1 \in \Im^{\mathfrak{K}_2}$, whereas $(\pi_0 \restriction A)[\mathfrak{A}] = \mathfrak{K}_2 \not\models (4.9)[x_0/0]]$.

Corollary 6.3. *Let* $\mathsf{P} \subseteq ([\mathsf{MSA} \cap \mathsf{B}]\mathsf{MSL})$ *be a pre-variety. Suppose* $\mathfrak{K}_2 \in | \notin \mathsf{P}(\supseteq (\varnothing|\{\mathfrak{DM}_4\}))$ *Then,* $\mathsf{K} \triangleq \mathrm{Si}_{\mathsf{P}}(\mathsf{NIP}(\cup([\mathsf{B}]\mathsf{NDML} \cap \mathsf{P}))) \subseteq \mathbf{I}(\{\mathfrak{B}_{2[0,1]}\} \cup ((\{\mathfrak{K}_2\}[\cap\varnothing])|\varnothing)(\cup[\mathsf{B}](\mathsf{A}|)\mathsf{DML})) \subseteq [\mathsf{B}](\mathsf{A}|)\mathsf{DML}$.

Proof. Consider any $\mathfrak{A} \in \mathsf{K} \subseteq \mathsf{P}$, in which case $|A| > 1$, and so we have the following $2(+1)[-1]$ exhaustive cases [but the first one]:

(1) $\mathfrak{A} \in \mathsf{TNIMSL}$,
in which case, by Lemma 5.33, $\mathfrak{K}_2$, being enbeddable into $\mathfrak{A}$ belongs to P, while, as $|A| > 1$, whereas $\mathfrak{K}_2$, being two-element, has no proper non-one-element subalgebra, by (2.8) and Corollary 4.7, there is some $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{K}_2)$, and so, by (2.5), h is injective, for $(\mathrm{img}\, h) = 2$, being two-element, is not a singleton.

(2) $\mathfrak{A} \in (\mathsf{NIP} \setminus (\mathsf{TNIMSL}[\cap\varnothing]))$,
in which case, by Lemma 5.3, $\mathfrak{B}_{2[,01]}$, being embeddable into $\mathfrak{A}$, belongs to P, while, as $\mathfrak{B}_{2[,01]}$ has no proper subalgebra, by Corollary 5.4, there is some $g \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, and so, by (2.5), g is injective, for $(\mathrm{img}\, g) = 2$, being two-element, is not a singleton.

(3) $\mathfrak{A} \notin (\mathsf{NIP} \cup (\mathsf{TNIMSL}[\cap\varnothing]))$,
in which case, by Lemma 5.3, $\mathfrak{K}_{3[,01]}$, being embeddable into $\mathfrak{A}$, belongs to P, while $\mathfrak{A} \in ([\mathsf{B}]\mathsf{NDML} \cap \mathsf{P})$. We prove that $\mathfrak{A} \in [\mathsf{B}]\mathsf{ADML}$, by contradiction. For suppose $\mathfrak{A} \notin [\mathsf{B}]\mathsf{ADML}$, in which case there are some $a, b \in A$ such that $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b) \not\leqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}b)$, and so, by (2.8) and Corollary 4.7, there are some $\mathfrak{B} \in \{\mathfrak{K}_{4:0[,01]}, \mathfrak{DM}_{4[,01]}\}$ and some $f \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $(\neg^{\mathfrak{B}}\neg^{\mathfrak{B}}f(a) \wedge^{\mathfrak{B}} \neg^{\mathfrak{B}}\neg^{\mathfrak{B}}f(b)) \not\leqslant^{\mathfrak{B}} (f(a) \vee^{\mathfrak{B}} \neg^{\mathfrak{B}}f(b))$. Then, $\neg^{\mathfrak{B}}\neg^{\mathfrak{B}}f(a) \neq f(a)$, in which case $\mathfrak{B} \neq \mathfrak{DM}_{4[,01]}$, and so $\mathfrak{B} = \mathfrak{K}_{4:0[,01]}$, while $\neg^{\mathfrak{B}}\neg^{\mathfrak{B}}f(b) \neq \neg^{\mathfrak{B}}f(b)$, in which case $f(b) \notin (3 \setminus 1)$, and so, by Lemma 5.40, $e \triangleq (f \circ h) \in \hom(\mathfrak{A}, \mathfrak{K}_{3[,01]})$, where $h \in \hom(\mathfrak{B}, \mathfrak{K}_{3[,01]}) \neq \varnothing$, whereas, since $\Im^{\mathfrak{K}_3} = \{1\}$, $\mathrm{img}\, e$, forming a subalgebra of $\mathfrak{K}_{3[,01]}$, is not a singleton, i.e., $e_*^{-1}[\Delta_3] = (\ker e) \neq A^2$. Thus, by (2.5), e is injective, in which case, by Corollary 4.7, $\mathfrak{A}$, being embeddable into $\mathfrak{K}_{3[,01]} \in \mathsf{ADML}$, belongs to this variety, and so Corollary 4.7, Lemma 6.2 and this contradiction complete the argument of this ()-optional case.

In this way, Corollary 4.7 completes the argument. □

Given any [finite] $\mathfrak{A} \in \mathsf{MSA}$, by (4.7) and (4.8), $(A \oplus 2) \triangleq ((A \times \{1\}) \cup \{\langle \bot^{\mathfrak{A}}, 0\rangle, \langle \top^{\mathfrak{A}}, 2\rangle\})$ [being finite] forms a subalgebra of $\mathfrak{A} \times \mathfrak{K}_{3,01}$, in which case $(\mathfrak{A} \oplus 2) \triangleq ((\mathfrak{A} \times \mathfrak{K}_{3,01}) \restriction (A \oplus 2)) \in \mathsf{MSA}$ [is finite].

Lemma 6.4. *Let* $\mathsf{P} \subseteq \mathsf{BMSL}$ *be a pre-variety and* $\mathfrak{A} \in \{\mathsf{I}\}\mathsf{MSA} \setminus \mathsf{KA})\langle$ *as well as* $f \in \hom(\mathfrak{A}, \mathfrak{K}_{3,01})\rangle$. *Suppose* $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, A^2\langle, \ker f\rangle\}$, *while* $\mathfrak{F} \triangleq (\mathfrak{A} \oplus 2) \in \mathsf{P} \not\ni \mathfrak{A}$, *whereas* $|A| \in \omega$. *Then,* $\mathfrak{F} \in (\mathrm{SI}_{\mathsf{P}}(\mathsf{P})\{\setminus \mathrm{Si}_{\mathsf{P}}(\mathsf{P})\})$.

Proof. In that case, since $(\pi_0{\restriction}F) \in \hom^{\mathrm{S}}(\mathfrak{F},\mathfrak{A})$, $\mathfrak{F} \notin \mathsf{KA} \ni \mathfrak{K}_{3,01}$ is not embeddable into $\mathfrak{K}^2_{3,01}$, and so, by Remark 4.2, the simplicity of $\mathfrak{K}_{3,01}$ (cf. Remark 4.3) as well as the non-optional version of Corollary 3.12 with $\mathfrak{B} = \mathfrak{C} = \mathfrak{K}_{3,01}$ and a unique $h \in (((A \times \{1\})\langle\cap\varnothing\rangle)\langle\cup\{f\}\rangle)$, $\mathfrak{F} \in \mathrm{SI}_{\mathsf{P}}(\mathsf{P})$. {And what is more, $\{\langle 0,\langle\bot^{\mathfrak{A}},0\rangle\rangle,\langle 1,\langle a,1\rangle\rangle,\langle 2,\langle\top^{\mathfrak{A}},2\rangle\rangle\}$, where $a \in \Im^{\mathfrak{A}} \neq \varnothing$, is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{F}$, in which case $\mathfrak{K}_{3,01} \in \mathsf{P}$, and so, as $e \triangleq (\pi_1{\restriction}F) \in \hom^{\mathrm{S}}(\mathfrak{F},\mathfrak{K}_{3,01})$, by (2.5), $(\ker e) \in (\mathrm{Co}_{\mathsf{P}}(\mathfrak{F}) \setminus \{\Delta_F, F^2\})$, for $|F| = (|A|+2) > 3 \neq 1$, as $\omega \ni |A| > 1$, because $\mathfrak{A}$ is idempotent.} □

Corollary 6.5. *Let* $\mathsf{P} \subseteq \mathsf{MSA}$ *be a relatively semi-simple pre-variety,* $\mathfrak{A} \in \mathsf{P}$, $\mathfrak{B} \in (\{\mathfrak{K}_{4:i,01} \mid i \in 2\} \cup \{\mathfrak{DM}_{4,01}\})$ *and* e *an embedding of* $\mathfrak{B}{\restriction}\Sigma^-_+$ *into* $\mathfrak{A}{\restriction}\Sigma^-_+$. *Then,* $\mathfrak{B} \in \mathsf{P}$.

Proof. By contradiction. For suppose $\mathfrak{B} \notin \mathsf{P}$, in which case e is not an embedding of $\mathfrak{B}$ into $\mathfrak{A}$, and so, by (4.7) and (4.8), both $e((\bot|\top)^{\mathfrak{B}}) \neq (\bot|\top)^{\mathfrak{A}}$. Then, by (4.7) and (4.8), $((\pi_0{\restriction}(B\times\{1\}))\circ e) \cup \{\langle\bot^{\mathfrak{B}},0,\bot^{\mathfrak{A}}\rangle,\langle\top^{\mathfrak{B}},2,\top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{B}\oplus 2$ into $\mathfrak{A}$, in which case $(\mathfrak{B}\oplus 2) \in \mathsf{P}$, and so Lemmas 5.40, 6.4, the simplicity of $\mathfrak{DM}_{4,01} \notin (\mathsf{MSA}\setminus\mathsf{KA})$ (cf., e.g., Remark 4.3), its finiteness and idempotence as well as those of $\mathfrak{K}_{4:(0|1)[,01]} \notin (\mathsf{MSA}\setminus\mathsf{KA})$ contradict to the relative semi-simplicity of P, as required. □

Theorem 6.6. *Any relatively semi-simple relatively subdirectly-representable (more specifically, "relatively semi-simple quasi-equational"/implicative) pre-variety* $\mathsf{P} \subseteq ([\mathsf{MSA}\cap\mathsf{B}]\mathsf{MSL})$ *is a sub-variety of* $[\mathsf{B}]\mathsf{ADML}$, *in which case it is* $\mathfrak{V}^{\vec{\varphi}}_{V_1|\Omega,\wp(\Omega)}$-*implicative, and so "*{*relatively*} ⟨*finitely-*⟩*semi-simple"//* $\lfloor\mathfrak{V}^{\vec{\varphi}}_{V_1\|\Omega,\wp(\Omega)}$-⌋*implicative sub-*{*quasi-//pre-*}*varieties of* $[\mathsf{MSA}\cap\mathsf{B}]\mathsf{MSL}$ *are exactly sub-varieties of* $[\mathsf{B}]\mathsf{ADML}$.

Proof. In that case, P is generated by $\mathsf{K} \triangleq \mathrm{Si}_{\mathsf{P}}(\mathsf{P})$. If there was some $\mathfrak{A} \in (\mathsf{P}\setminus(\mathsf{NI}[\mathsf{B}]\mathsf{MSL}\cup[\mathsf{B}]\mathsf{NDML}))$, then, by Lemma 5.30 [and Corollary 6.5], $\mathfrak{K}_{4:i[,01]}$ would be in P, for some $i \in 2$, contrary to the relative semi-simplicity of P and Corollary 6.1. Hence, $\mathsf{K} \subseteq \mathsf{P} \subseteq (\mathsf{NI}[\mathsf{B}]\mathsf{MSL}\cup[\mathsf{B}]\mathsf{NDML})$, in which case, by Corollary 6.3, $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{ADML}$, and so $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{ADML}$. Consider the following complementary cases:

- $\mathsf{K} = \varnothing$,
  in which case $\mathsf{P} = [\mathsf{B}]\mathsf{OMSL}$.
- $\mathsf{K} \neq \varnothing$.
  Consider the following complementary subcases:
  - $\mathsf{K} \subseteq \mathsf{NI}[\mathsf{B}]\mathsf{MSL}$.
    in which case, by Footnote 1 and Lemma 6.3, $\mathsf{K} = \mathbf{I}((\mathsf{P}\cap\{\mathfrak{B}_{2[,01]}\})\cup((\mathsf{P}\cap\{\mathfrak{K}_2\})[\cap\varnothing]))$, and so, by Corollary 4.7, $\mathsf{P} = ([\mathsf{B}](\mathsf{A}|)\mathsf{BL}\|([\mathsf{BOMSL}\cap\mathsf{B}]\mathsf{TNIMSL})|[\mathsf{B}]\mathsf{OMSL})$, whenever $\mathfrak{K}_2 \in | \notin \mathsf{P} \ni \| \not\ni \mathfrak{B}_{2[,01]}$.
  - $\mathsf{K} \nsubseteq \mathsf{NI}[\mathsf{B}]\mathsf{MSL}$.
    Consider the following complementary subcases:
    - $\mathsf{K} \subseteq ([\mathsf{B}]\mathsf{KSL}\cup\mathsf{NI}[\mathsf{B}]\mathsf{MSL})$,
      in which case $\mathsf{IK} \subseteq [\mathsf{B}]\mathsf{AKL}$, and so, by Lemma 6.3, $\mathsf{P} \subseteq [\mathsf{B}]\mathsf{AKL}$. Conversely, take any $\mathfrak{A} \in \mathsf{IK} \neq \varnothing$, in which case, by Lemma 5.3, $\mathfrak{K}_{3[,01]} \in \mathsf{P}$, so, by Corollary 4.7 and Lemma 6.2, $\mathsf{P} = [\mathsf{B}](\mathsf{A}|)\mathsf{KL}$, whenever $\mathfrak{K}_2 \in | \notin \mathsf{P}$.
    - $\mathsf{K} \nsubseteq ([\mathsf{B}]\mathsf{QSKSL}\cup\mathsf{NI}[\mathsf{B}]\mathsf{QSMSL}$.
      Take any $\mathfrak{B} \in (\mathsf{K}\setminus([\mathsf{B}]\mathsf{QSKSL}\cup\mathsf{NI}[\mathsf{B}]\mathsf{QSMSL})) \neq \varnothing$, in which case, by Lemma 5.28 [and Corollary 6.5], $\mathfrak{DM}_{4[,01]} \in \mathsf{P}$, and so, by Corollary 4.7 and Lemma 6.2, $\mathsf{P} = [\mathsf{B}](\mathsf{A}|)\mathsf{DML}$, whenever $\mathfrak{K}_2 \in | \notin \mathsf{P}$.

This, by Corollary 4.7 (and Remark/Corollary 2.4/3.4), completes the proof. □

This supersedes the reservation "quasi-equational/finitely" in Corollary 4.7 for the unbounded case.

7. Conclusions

Perhaps, the most acute problems remained open concern the lattice of quasi-varieties of *all* MS lattices. First of all, taking Subsection 5.7 and Corollary 5.16 into account, it would be especially valuable to find out whether varieties of MS lattices including $\mathsf{ADML} \supseteq \mathsf{ABL}$ — NDML and $[(\mathsf{PS})|(\mathsf{AQS})]\mathsf{MSL}$ — are Q-universal. Likewise, it would be equally important to learn whether the lattices of other varieties subsumed by neither QSMSL nor AQSKSL — NKL and $[\mathsf{PS}](\mathsf{W})\mathsf{KSL} \supseteq \mathsf{NKL}$ — are finite. Such equally concerns extension of Section 6 beyond MS algebras in the bounded case. After all, an interesting (though purely methodological) point remained open is to find equational proofs (like that of (4.18)) of the rather curious inclusions such as

$$[\mathsf{B}/]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{PSMS}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{WKMS}(\mathsf{L}[/\mathsf{A}])$$

and $[\mathsf{B}/]\mathsf{QSWKS}(\mathsf{L}[/\mathsf{A}]) \subseteq [\mathsf{B}/]\mathsf{QSKS}(\mathsf{L}[/\mathsf{A}])$ as well as both

$$(\mathsf{NIMR}[\mathsf{B}]\mathsf{QSMSL} \cup \mathsf{MR}[\mathsf{B}](\mathsf{QS})\mathsf{KSL}) \subseteq (\mathsf{NIMR}[\mathsf{B}]\mathsf{QSMSL} \cup [\mathsf{B}]\mathsf{KL})$$

and $\mathsf{NQSRVQSMSL} \subseteq \mathsf{QSRVQSMSL}$, just ensuing from Corollaries 4.7, 5.14 and Theorem 5.25. Likewise, the fact that abstract non-trivially-hereditary subclasses of $\mathrm{SI}([\mathsf{B}]\mathsf{MSL})$ are its relative sub-varieties looks too occasional to refrain from raising the question whether such is the case, in general, for arbitrary varieties like $[\mathsf{B}]\mathsf{MSL}$ (e.g., disjunctive finitely-generated ones of lattice expansions with REDPC {being the pre-varieties generated by the underlying algebras of finite matrices with equality determinant and truth predicates being prime filters of their underlying algebras}).

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