The Complete Proof of the Riemann Hypothesis

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Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show there is a contradiction just assuming the possible smallest counterexample $n > 5040$ of the Robin inequality. In this way, we prove that the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

Keywords: Riemann hypothesis, Robin inequality, sum-of-divisors function, prime numbers
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [1].*

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_m$ is called an Hardy-Ramanujan integer [2]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

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Theorem 1.2. *If n is superabundant, then n is an Hardy-Ramanujan integer [3].*

Theorem 1.3. *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [4].*

We prove the nonexistence of such counterexample and therefore, the Riemann Hypothesis is true.

2. Proof of Main Theorems

Let $n = \prod_{i=1}^{s} q_i^{e_i}$ be a factorisation of n, where we ordered the primes q_i in such a way that $e_1 \geq e_2 \geq \cdots \geq e_s$. We say that $\overline{e} = (e_1, \ldots, e_s)$ is the exponent pattern of the integer n [2]. Note that $\prod_{i=1}^{s} p_i^{e_i}$ is the minimal number having exponent pattern $\overline{e}$ when $p_1 = 2, p_2 = 3, \ldots, p_s$ denote the first s consecutive primes and $e_1 \geq e_2 \geq \cdots \geq e_s$. We denote this (Hardy-Ramanujan) number by $m(\overline{e})$ [2].

Theorem 2.1. *Let $\prod_{i=1}^{m} q_i^{e_i}$ be the representation of n as a product of the primes $q_1 < \cdots < q_m$ with natural numbers as exponents $e_1, \ldots, e_m$. We obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold.*

Proof. According to the theorems 1.2 and 1.3, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $e_1 \geq e_2 \geq \cdots \geq e_m$ since $n > 5040$ should be an Hardy-Ramanujan integer. Let $\overline{e}$ denote the factorisation pattern of $n \times q_m$. Based on the result of the article [5], the value $n \times q_m$ cannot be a square full number [2]. Therefore $n \times q_m > m(\overline{e})$ and consequently, $n > \frac{m(\overline{e})}{q_m}$. Thus, we have that $\mathsf{Robins}(\frac{m(\overline{e})}{q_m})$ holds, because of $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. We know that $f(p^e) > f(q^e)$ if $p < q$ [2]. In this way, we would have that $f(\frac{m(\overline{e})}{q_m}) > f(n)$ since $f(q_i^2) > f(q_i) \times f(q_m)$ for some positive integer $1 \leq i < m$. Certainly, we have that

$$\frac{f(q_i^2)}{f(q_i)} = \frac{q_i^3 - 1}{q_i^2 \times (q_i - 1)} \times \frac{q_i}{q_i + 1} = \frac{q_i^3 - 1}{q_i^3 - q_i}. \tag{1}$$

Let's define $\omega(n)$ as the number of distinct prime factors of n [2]. From the article [5], we know that $\omega(n) \geq 969672728$ and the number of primes lesser than q_m which have the exponent equal to 1 in n is approximately

$$\omega(n) - \frac{\omega(n)}{14} = \frac{13 \times \omega(n)}{14} \geq \frac{13 \times 969672728}{14} > 900410390.$$

In this way, there exists a positive integer $1 \leq i < m$ such that

$$\frac{f(q_i^2)}{f(q_i)} = \frac{q_i^3 - 1}{q_i^3 - q_i} \geq f(q_{i+900000000}) > f(q_m)$$

where we could have that $q_i^2 \nmid n$, $q_i \mid n$, $q_{i+900000000} \mid n$ and $q_i^2 \mid \frac{m(\overline{e})}{q_m}$. Finally, we have that

$$f(n) < f(\frac{m(\overline{e})}{q_m}) < e^{\gamma} \times \log\log \frac{m(\overline{e})}{q_m} < e^{\gamma} \times \log\log n.$$

However, this a contradiction with our initial assumption. To sum up, we obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. □

Theorem 2.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof. Due to the theorem 2.1, we can assure there is not any natural number $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold. □

Theorem 2.3. *The Riemann Hypothesis is true.*

Proof. This is a direct consequence of theorems 1.1 and 2.2 □

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