# A Collection of Service Time Distributions Parameters Study and Impact in M|G|∞ System Busy Period and Busy Cycle Length Distributions

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# A COLLECTION OF SERVICE TIME DISTRIBUTIONS PARAMETERS STUDY AND IMPACT IN M|G|∞ SYSTEM BUSY PERIOD AND BUSY CYCLE LENGTH DISTRIBUTIONS

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## ABSTRACT

We present the problems that arise when calculating the moments of service time probability distributions for which the M|G|∞ queue system busy period and busy cycle-an idle period followed by a busy period-length probability distributions become very easy to study and show how to overcome them. We also, calculate the renewal function, the "peakedness," and the "modified peakedness" for the M|G|∞ busy period and busy cycle length in the case of those service time distributions.

**Keywords**: service time, collection, probability distribution, moment, M|G|∞ queue

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## 1. INTRODUCTION

When, in the M|G|∞ queue system, the service time is a random variable with a distribution function belonging to the collection

$$G(t) = 1 - \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)t} - 1\right) + \lambda}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1},$$

$$0 \leq p < 1, \tag{1.1}$$

($\rho = \lambda\alpha$[1], being $\alpha$ the mean value associated with $G(t)$ and $\lambda$ the customers'arrivals rate) the busy period length probability distribution, which distribution function is:

$$B^{\beta}(t) = 1 - \frac{\lambda + \frac{\lambda p + \beta}{1-p}}{\lambda}(1-e^{-\rho})e^{-e^{-\rho}\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)t}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq$$

$$p < 1 \tag{1.2}$$

is an exponential distribution with an atom at the origin. Moreover, the busy cycle length probability distribution, which distribution function is:

$$Z^{\beta}(t) = 1 - \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda - e^{-\rho}\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)} e^{-e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t} + \frac{\frac{\lambda p+\beta}{1-p}}{\lambda - e^{-\rho}\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)} e^{-\lambda t}, t \geq 0, -\lambda \leq \beta \leq$$

$$\frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1 \tag{1.3}$$

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[1] The parameter $\rho$ is the traffic intensity.

is a mixture of two exponential distributions, see [1, 2]. However, although it is so easy to study the busy period and the busy cycle in this situation, it is a quite complex task to compute the service time moments.

Some results, precisely about the moment's calculation of random variables with distribution functions given by this collection are given.

In the end, we present formulae that give the busy cycle renewal function, the "peakedness," and the "modified peakedness" to the busy period and the busy cycle of the M|G|∞ system for those service time distributions, see [3-6]. We shall see how the formulae for these parameters in this case are quite simple contrarily to what is the usual.

This work stands on the information presented in [7], corrected, generalized, and updated.

## 2. CALCULATION OF MOMENTS

Be $G(t), t \geq 0$ a positive random variable distribution function, and $g(t) = \frac{dG(t)}{dt}$ the associated probability density function.

The differential equation $(1-p)\frac{g(t)}{1-G(t)} - \lambda p - \lambda(1-p)G(t) = \beta$, where $\lambda > 0$ and $-\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1$ has the solution in expression (1.1), see again [7].

If, in (1.1), $G_i(t)$ is the solution associated to $\rho_i, i = 1, 2, 3, 4$ it is easy to check that

$$\frac{G_4(t)-G_2(t)}{G_4(t)-G_1(t)}\frac{G_3(t)-G_1(t)}{G_3(t)-G_2(t)} = \frac{e^{-\rho_4}-e^{-\rho_2}}{e^{-\rho_4}-e^{-\rho_1}}\frac{e^{-\rho_3}-e^{-\rho_1}}{e^{-\rho_3}-e^{-\rho_2}} \tag{2.1}$$

as it had to happen since it is a Riccati equation, see [8].

And calculating,

$$\int_0^\infty [1-G(t)]dt = \int_0^\infty \frac{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda}dt =$$

$$\frac{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{1}{e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+1}dt =$$

$$\frac{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}-e^{-\rho}\,e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}+e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}dt =$$

$$\frac{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}+(1-e^{-\rho})e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}dt =$$

$$\frac{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\frac{-1}{(1-e^{-\rho})\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}\left[\ln\left(e^{-\rho}+(1-e^{-\rho})e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\right]_0^\infty = -\frac{1}{\lambda}\ln e^{-\rho} = \frac{\rho}{\lambda} = \alpha.$$

As it had to be because we are dealing with a positive random variable. The probability density function associated to $G(t)$ given by (1.1) is

$$g(t) = \frac{(1-e^{-\rho})e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2 e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\lambda\left[e^{-\rho}+(1-e^{-\rho})\,e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}, t>0, -\lambda \le \beta \le \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1},$$

$0 \leq p < 1$ (2.2)

So,

$$\int_0^\infty t^n g(t)dt = \frac{(1-e^{-\rho})e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2}{\lambda}\int_0^\infty t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+(1-e^{-\rho})\, e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\ . \tag{2.3}$$

But, $\int_0^\infty t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+(1-e^{-\rho})\, e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt \geq \int_0^\infty t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt =$

$\frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, \beta \neq -\lambda$. And $\int_0^\infty t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+(1-e^{-\rho})\, e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt \leq$

$e^{2\rho}\int_0^\infty t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt = \frac{e^{2\rho}}{\lambda+\frac{\lambda p+\beta}{1-p}}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, \beta \neq -\lambda.$

Therefore, calling $T$ the random variable with distribution function $G(t)$, and having in mind (2.3):

$$\frac{(1-e^{-\rho})e^{-\rho}}{\lambda}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}} \leq E[T^n] \leq \frac{e^{\rho}-1}{\lambda}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}}, -\lambda < \beta \leq$$

$\frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1, n = 1,2,\ldots$ (2.4).

*Notes:*

- The expression (2.4), giving bounds for $E[T^n], n = 1,2, \dots$ guarantees its existence,
- For $n = 1$ the expression (2.4) is useless since $E[T] = \alpha$. Note, curiously, that the upper bound is $\frac{e^\rho - 1}{\lambda}$, the M|G|∞ system busy period mean value[2],
- For $n = 2$, subtracting to both bounds $\alpha^2$, it is possible to get from expression (2.3) bounds for $VAR[T]$,
- For $\beta = -\lambda, \mathrm{E}[\mathrm{T}^n] = 0, n = 1,2,\dots,$ evidently.

See, however, that (1.1) is writable like:

$$G(t) = \frac{1 + \frac{\frac{\lambda p + \beta}{1-p}}{\lambda}(1 - e^\rho)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}}{1 - (1 - e^\rho)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda(1 - pe^\rho)}{e^\rho - 1}, 0 \leq p < 1, \qquad (2.5)$$

And, for $\rho < \ln 2$,

$$G(t) = \left(1 + \frac{\frac{\lambda p + \beta}{1-p}}{\lambda}(1 - e^\rho)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}\right)\sum_{k=0}^{\infty}(1 - e^\rho)^k e^{-k\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda(1 - pe^\rho)}{e^\rho - 1}, 0 \leq p < 1, \qquad (2.6)$$

After (2.6) it is an easy task to obtain the random variable $T$ Laplace transform, and the consequent formulae for the moments. Indeed:

- For $\rho < \ln 2$,

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[2] Insensible to the service time probability distribution.

$$E[T^n] = -\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right) n! \sum_{k=1}^{\infty} \frac{(1-e^{\rho})^k}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0$$

$$\leq p < 1, n = 1,2,\ldots \tag{2.7}$$

*Notes:*

- $E[T] = -\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right)\sum_{k=1}^{\infty}\frac{(1-e^{\rho})^k}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)} = \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1-e^{\rho}}{k} = \frac{1}{\lambda}\ln e^{\rho} = \alpha$
- For $n \geq 2$, having to consider only a finite number of parcels in the infinite sum, call M this number. To get an error lesser than $\varepsilon$ it must be fulfilled simultaneously

a. $M > \frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}} - 1,$

b. $M > \log_{(\varepsilon^{\rho}-1)} \frac{\varepsilon e^{\rho}\lambda}{n!\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}$

Therefore, it is evident now that this distributions collection moment's computation is a complex task. This was already true for the study in [9] where the results presented are a situation of these ones: $p = 0$.

To consider the approximation

$$E_m^n = \sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^n\left[G\left(\frac{k}{m}\right) - G\left(\frac{k-1}{m}\right)\right], -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0$$

$$\leq p < 1, n = 1,2,\ldots \tag{2.8}$$

may be helpful since $\lim_{m\to\infty} E_m^n = E[T^n], n = 1,2, \ldots$ that allow the moments' numerical computation, see [10].

## 3. Busy Cycle Renewal Function Calculation

The busy cycle renewal function, of the M|G|∞ queue, at $t$, designated $R(t)$, gives the mean number of busy periods that begin in $[0, t]$, and its expression is, see again [3]:

$$R(t) = e^{-\lambda \int_0^t [1-G(v)]dv} + \lambda \int_0^t e^{-\lambda \int_0^u [1-G(v)]dv} du \,. \tag{3.1}$$

If the service time is a random variable with distribution function given by a member of the collection (1.1):

$$R(t) = e^{-\rho}(1 + \lambda t) + (1 - e^{-\rho}) \frac{\lambda p+\beta}{\lambda+\beta} e^{-\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)t} + (1 - e^{-\rho}) \frac{\lambda p+\beta}{\lambda+\beta}, -\lambda \le \beta \le \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \le p < 1, n = 1,2, \ldots \tag{3.2}$$

For $p = 0$ we obtain the result presented in [3].

## 4. The "Peakedness" and the "Modified Peakedness" Calculations

The M|G|∞ queue busy period "peakedness" is the Laplace Transform of its time length[3] at $\frac{1}{\alpha}$,

$$p_i = \bar{B}\left(\frac{1}{\alpha}\right) = 1 + \lambda^{-1}\left(\frac{1}{\alpha} - \frac{1}{\int_0^\infty e^{-\frac{1}{\alpha}t - \lambda \int_0^t [1-G(v)]dv} dt}\right) \tag{4.1}$$

[3] Called $\overline{B}(s)$.

[4, 5]. This parameter characterizes the distribution of the busy period time length. It contains information on all its moments. For the collection of service time distributions (1.1) the “peakedness” is

$$p_i = \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{\lambda(e^{-\rho}(\rho+\alpha\beta)+1-p)}, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1. \quad (4.2)$$

In [4, 5] another measure is introduced, the “modified peakedness” got after the “peak” taking out the terms that are permanent for the busy period in different service distributions and dividing for the common part. Calling $q_i$:

$$q_i = p_i \frac{\rho}{e^{\rho}-\rho-1} + 1 \quad (4.3)$$

and so, for the distributions given by collection (1.1):

$$q_i = \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p+\beta}{\lambda(e^{-\rho}(\rho+\alpha\beta)+1-p)} \frac{\rho}{e^{\rho}-\rho-1} + 1, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1 \quad (4.4)$$

For the busy cycle of the M|G|∞ queue, analogously, define also the “peakedness” [5], now called $p_i'$ and for the service distributions given by the collection (1.1) it is

$$p_i' = \alpha \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{(\rho+1)(e^{-\rho}(\rho+\alpha\beta)+1-p)}, -\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \leq p < 1 \quad (4.5)$$

And the “modified peakedness,” now called $q_i'$, given by

$$q_i' = p_{i'} \frac{\rho}{e^{\rho}-\rho} + 1 . \quad (4.6)$$

For the service distributions given by the collection (1.1) it is

$$q_i' = \alpha \frac{e^{-\rho}(\lambda+\beta)(\rho+1) - \lambda p - \beta}{(\rho+1)(e^{-\rho}(\rho+\alpha\beta)+1-p)} \frac{\rho}{e^{\rho}-\rho} + 1,$$
$$-\lambda \le \beta \le \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}, 0 \le p < 1 \quad (4.7)$$

## CONCLUSION

The study of the M|G|∞ queue transient probabilities behavior as time functions leads to the consideration of a Riccati equation. Its solution, subject to specific constraints, is a collection of service time distributions for which both the busy period and the busy cycle have rather simple probability distributions for the respective time lengths. These distributions are either an exponential, or an exponential with an atom at the origin or a mixture of two exponentials.

Either this is the most important since the distributions related to the busy period or the busy cycle of queues are extremely complex, even analytically intractable, usually given by series of convolutions.

The approach followed in this work generalize the results shown in [11].

In M|G|∞ queue practical applications, see for instance [12-15], the probabilistic study of the length of the busy period and of the busy cycle is of great importance. Note, for example, that a practical embodiment of the presence of infinite servers lies in the requirement that when a customer arrives, a server must be immediately available. It is therefore necessary to foresee how long the busy period is to know how long the servers should be in the prevention.

For more information on these subjects see, for instance [16-19].

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