Chang-Mundici construction of an enveloping unital lattice-group of a BL-algebra

Celestin Lele[1] and Jean B Nganou[2]

[1] Department of Mathematics and Computer Science, University of Dschang, Cameroon
celestinlele@yahoo.com
[2] Department of Mathematics, University of Oregon, Eugene, OR 97403
nganou@uoregon.edu

Abstract

For any BL-algebra L, we construct an associated lattice ordered Abelian group G_L that coincides with the Chang's ℓ-group of an MV-algebra when the BL-algebra is an MV-algebra. We prove that the Chang's group of the MV-center of any BL-algebra L is a direct summand in G_L. We also find a direct description of the complement $\mathfrak{S}(L)$ of the Chang's group of the MV-center in terms of the filter of dense elements of L. Finally, we compute some examples of the group G_L.

1 Introduction

It is well known that the category of MV-algebras is equivalent to that of ℓ-groups with strong unit [2]. This equivalence, which depends in large part on the natural algebraic addition of MV-algebras has been an essential tool in the study of MV-algebras. On the other hand, MV-algebras are BL-algebras whose negations are involutions. Given that a proper algebraic addition is now available in BL-algebras [5], it is natural to consider some of the constructions of MV-algebras that rely on the addition within the BL-algebras framework. Undoubtably, the Chang-Mundici's ℓ-group of an MV-algebra is one of the most important such constructions [1, 2].

The main goal of the present work is to extend this construction to BL-algebras, by associating to each BL-algebra an ℓ-group with strong unit. More specifically, we introduce good sequences in any BL-algebra L and show that their set forms a commutative monoid $(M_L, +, 0)$. Mimicking the general construction due to Grothendieck of an Abelian group from a commutative monoid, we obtain a lattice ordered Abelian group with strong unit G_L, which we shall refer to as the Chang's ℓ-group of the BL-algebra. We show that the Chang's group of the MV-center of L is a direct summand in G_L, and that its complement $\mathfrak{S}(L)$ is an ideal of G_L. For BL-chains, we obtain that G_L is isomorphic to the lexicographic product of the Chang's group of the MV-center and $\mathfrak{S}(L)$. It turns out that the ℓ-group $\mathfrak{S}(L)$ can be directly constructed from the filter of dense elements of the BL-algebra L, thus generalizing the relationship between BL-algebras and negative cones of ℓ-groups [4].

2 Addition and Good Sequences in BL-algebras

The following natural commutative operation $+$ was introduced in every BL-algebra [5]: For every $x, y \in L$,

$$x + y := (\bar{x} \to y) \wedge (\bar{y} \to x)$$

This addition is commutative, associative, has 0 for identity, and 1 as absorbing.

Definition 2.1. A BL-chain L is called *of cancellative type* if for every $x, y, z \in L$,

$$x + y = x + z \text{ and } x \otimes y = x \otimes z \text{ imply } y = z.$$

A BL-algebra is called of cancellative type if it is a subdirect product of BL-chains of cancellative type.

Examples of BL-algebras of cancellative type include MV-algebras, and product algebras.

Lemma 2.2. *In every BL-algebra L, the following equation holds.*

$$(x \otimes y) + ((x + y) \otimes z) = (x \otimes z) + ((x + z) \otimes y)$$

The following definitions are motivated by similar concepts in MV-algebras.

Definition 2.3. Let L be a BL-algebra. A sequence $\mathbf{a} := (a_1, a_2, \ldots, a_n, \ldots)$ of elements of L is called a good sequence if for all i, $a_i + a_{i+1} = a_i$, and there exists an integer n such that $a_r = 0$ for all $r > n$. In this case, instead of writing $\mathbf{a} = (a_1, a_2, \ldots, a_n, 0, 0 \ldots)$, we will simply write $\mathbf{a} = (a_1, a_2, \ldots, a_n)$

Note that if, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a good sequence, then $(\underbrace{1, \ldots, 1}_{m\ 's\ 1}, a_1, a_2, \ldots, a_n)$ is again a good sequence. The later will be denoted by $(1^m, a_1, a_2, \cdots, a_n)$.
We define the addition of sequences as follows:

Definition 2.4. Let L be a BL-algebra and let $\mathbf{a} = (a_1, a_2, \ldots, a_n, \ldots)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n, \ldots)$ be sequences in L. We define the sum of $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} + \mathbf{b} = (c_1, c_2, \ldots, c_n, \ldots)$ where $c_i := a_i + (a_{i-1} \otimes b_1) + \cdots + (a_1 \otimes b_{i-1}) + b_i$.

Proposition 2.5. *For every BL-algebra L, $(M_L, +, (0))$ is an Abelian monoid satisfying: for every $\mathbf{a}, \mathbf{b} \in M_L$,*

$$\mathbf{a} + \mathbf{b} = (0) \quad \textit{implies} \quad \mathbf{a} = \mathbf{b} = (0)$$

Using the Grothendieck construction of a group from a commutative monoid, we define on $M_L \times M_L$ the relation $\sim$ by $(\mathbf{a}, \mathbf{b}) \sim (\mathbf{c}, \mathbf{d})$ if and only if there exists $\mathbf{k} \in M_L$ such that $\mathbf{a} + \mathbf{d} + \mathbf{k} = \mathbf{b} + \mathbf{c} + \mathbf{k}$. Then, it is easy to see that $\sim$ is an equivalence relation on $M_L \times M_L$. Let $G_L = M_L \times M_L / \sim$ be the factor set of $M_L \times M_L$ by $\sim$. If we denote the equivalence class of $(\mathbf{a}, \mathbf{b})$ by $[\mathbf{a}, \mathbf{b}]$, one has a well-defined operation $+$ on G_L given by $[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}]$. Moreover, $(G_L, +, [(0), (0)])$ is an Abelian group where $-[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}]$. We define $\leq$ on the monoid $(M_L, +)$ by: $\mathbf{a}, \mathbf{b} \in M_L$; $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all i. It is clear that $\leq$ is a partial order on M_L.

Proposition 2.6. *Let $\preceq$ be the relation defined on G_L by $[\mathbf{a}, \mathbf{b}] \preceq [\mathbf{c}, \mathbf{d}]$ if and only if there exists $\mathbf{k} \in M_L$ such that $\mathbf{a} + \mathbf{d} + \mathbf{k} \leq \mathbf{b} + \mathbf{c} + \mathbf{k}$.*
Then $\preceq$ is a partial order relation on G_L that is translation invariant.

Theorem 2.7. *For every BL-algebra L, the po-group G_L is a lattice ordered group where,*

$$[\mathbf{a}, \mathbf{b}] \vee [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \vee (\mathbf{b} + \mathbf{c}), \mathbf{b} + \mathbf{d}] \quad \textit{and} \quad [\mathbf{a}, \mathbf{b}] \wedge [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \wedge (\mathbf{b} + \mathbf{c}), \mathbf{b} + \mathbf{d}]$$

Furthermore, $u_L := [(1), (0)]$ is a strong unit of the ℓ-group G_L.

Remark 2.8. The standard proof of Theorem 2.7 for MV-algebras relies on the fact that if A is an MV-algebra, then $\mathbf{a} \leq \mathbf{b}$ in M_A if and only if there exists $\mathbf{c} \in M_A$ such that $\mathbf{b} = \mathbf{a} + \mathbf{c}$ [3, Prop. 2.3.2]. However, this condition does not hold for general BL-algebras, not even for those of cancellative type.

Theorem 2.9. *Let L be a BL-algebra whose MV-center is denoted by A with Chang-Muncidi's group G_A. Then, there exists an ideal $\mathfrak{S}(L)$ of G_L such that*

$$G_L = G_A \oplus \mathfrak{S}(L)$$

In particular, the Chang's ℓ-group G_A of A is a direct summand of G_L.

Corollary 2.10. *Let L be a BL-algebra whose MV-center is A, then;*
There exists an ordered-preserving group isomorphism (o-isomorphism) from G_L onto $G_A \times_{lex} \mathfrak{S}(L)$ (the lexicographic product), where $\mathfrak{S}(L)$ is the lattice ordered group above.

For BL-algebras of cancellative type, several aspects of the construction can be simplified as we will see next.

Proposition 2.11. *If L is a BL-algebra of cancellative type, then $(M_L, +)$ is a cancellative monoid, that is for every good sequences $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in L such that $\mathbf{a}+\mathbf{c} = \mathbf{b}+\mathbf{c}$, then $\mathbf{a} = \mathbf{b}$.*

Corollary 2.12. *For every BL-algebra L of cancellative type, $(M_L, +, \leq)$ is submonoid of the positive cone of G_L and the restriction of the order of G_L to M_L coincides with $\leq$.*

Corollary 2.13. *A BL-algebra of cancellative type is an MV-algebra if and only if $\mathfrak{S}(L) = 0$.*

3 Examples and alternate descriptions

This section is devoted to computations of the Chang's ℓ-group of some of the most important BL-algebras.

Remark 3.1. Suppose that L is a BL-chain, then every element $[\mathbf{a}, \mathbf{b}] \in G_L$ has the form $[(1^p, a), (1^q, b)]$ with $p, q \geq 0$, $a \neq 0$ and $b \neq 0$. In addition, for a BL-chain L, the ℓ-group $\mathfrak{S}(L)$ can be described as follows. $\mathfrak{S}(L) = \{[(a), (b)] : a, b \in L \text{ and } \overline{\overline{a}} = \overline{\overline{b}}\} = \{[(a), (b)] : a, b \in L \text{ and } \overline{a} = \overline{b}\}$.

Proposition 3.2. *The Chang's ℓ-group of any linearly ordered Gödel algebra is isomorphic to the o-group $\mathbb{Z}$.*

Lemma 3.3. *1. If L is linearly ordered BL-algebra, then G_L is an o-group.*
2. If L is a BL-algebra of cancellative type so that G_L is an o-group, then L is a BL-chain.

Corollary 3.4. *For every BL-chain L,*

$$G_L \cong G_A \times_{lex} \mathfrak{S}(L)$$

Proposition 3.5. *Let $L = [0, 1]$ with the product structure. Then G_L is isomorphic to $\mathbb{Z} \times_{lex} \mathbb{R}_+$.*

Proposition 3.6. *If L is a product BL-chain, then the Chang's ℓ-group of L is naturally isomorphic to $\mathbb{Z} \times_{lex} \mathfrak{H}(L)$, where $\mathfrak{H}(L)$ is the o-group whose negative cone is L.*

Proposition 3.7. *For every BL-algebra L of finite order, $G_L = G_{MV(L)}$.*

Corollary 3.8. *A BL-algebra of finite order is of cancellative type if and only if it is an MV-algebra.*

Corollary 3.9. *For every Gödel-Dummit algebra H, G_H is equal to the Chang's group of its Boolean center $MV(H)$.*

As observed in Theorem 2.9, the Chang's group of a BL-algebra is built up of two essential pieces: the Chang's group of its MV-center and the ideal $\mathfrak{S}(L)$. Therefore, in order to understand the Chang's group of a general BL-algebra, it is enough to understand the group $\mathfrak{S}(L)$. To each filter F of a BL-algebra L, one can associate an Abelian group G_F whose elements are of the form $[a, b]$ with $a, b \in F$ and $[a, b] = [x, y]$ if and only if there exists $t \in F$ such that $a \otimes y \otimes t = x \otimes b \otimes t$. In addition $(G_F, +, -, 0)$ is an Abelian group where:

$$[a, b] +_G [c, d] = [a \otimes c, b \otimes d], \; -[a, b] = [b, a], 0_G = [1, 1]$$

On the other hand, consider on G_F the order $\leq$ defined by $[a, b] \leq [x, y]$ if and only if there exists $t \in F$ such that $a \otimes y \otimes t \leq x \otimes b \otimes t$.

Proposition 3.10. *Let L be a BL-algebra L and F a filter of L, then $\mathfrak{G}_F := (G_F, +, -, 0, \leq)$ is a lattice-ordered Abelian group with:*

$$[a, b] \vee [x, y] = [(a \otimes y) \vee (x \otimes b), b \otimes y] \;\; \textit{and} \;\; [a, b] \wedge [x, y] = [(a \otimes y) \wedge (x \otimes b), b \otimes y]$$

For every BL-algebra L, consider the set $F(L)$ of dense elements in L, that is:

$$F(L) = \{x \in L : \bar{x} = 0\}$$

Then it is readily seen that $F(L)$ is a filter of L. For BL-algebras of cancellative type, the following result provides a direct description of the group $\mathfrak{S}(L)$ in terms of group of the filter $F(L)$.

Proposition 3.11. *For every BL-algebra of cancellative type L,*

$$\mathfrak{G}_{F(L)} \cong \mathfrak{S}(L)$$

Since there are important classes of algebras generalizing BL-algebras (e.g. MTL-algebras, residuated lattices), our future work will investigate the generalization of the constructions obtained here to these classes.

References

[1] R. Cignoli, D. Mundici, An elementary proof of Chang's completeness theorem for the infinite-valued calculus of Lukasiewicz, *Studia Logica*, (special issue on the treatment of uncertain information)**58**(1998) 79-97.

[2] R. Cignoli, D. Mundici, An elementary presentation of the equivalence between MV-algebras and ℓ-groups with strong units, *Studia Logica*, special issue on Many-valued logics **61**(1998) 49-64.

[3] R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic foundations of many-valued reasoning, *Kluwer Academic, Dordrecht*(2000).

[4] Cignoli, R., A. Torrens, An algebraic analysis of product logic, *Mult.-Valued Log.* **5** (2000) 45-65.

[5] C. Lele, J.B. Nganou, Algebraic Addition in BL-algebras, submitted