Dcpo models of T_1 spaces

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A poset model of a topological space X is a poset P together with a homeomorphism $\phi : X \longrightarrow \mathrm{Max}(P)$ ($\mathrm{Max}(P)$ is the subspace of the Scott space ΣP consisting of maximal points of P). In [11] (also in [2]), it was proved that every T_1 space has a bounded complete algebraic poset model. It is, however still unclear whether each T_1 space has a dcpo model. In this paper we give a positive answer to this problem. In section 1, we show that every T_1 space has a dcpo model. In section 2, we prove that a T_1 space is sober if and only if its dcpo model constructed in section 1 is a sober dcpo. These results provide us with a method to construct non-sober dcpos from any non-sober T_1 spaces. In section 3, for some special spaces we construct a more concrete dcpo model.

1 Dcpo models of T_1 spaces

Theorem 1. *[11] Every T_1 space has a bounded complete algebraic poset model.*

Remark 1. Let X be a T_0 space and $\mathcal{A}$ be the set of all filters of open sets of X that has a nonempty intersection. Then $(\mathcal{A}, \subseteq)$ is a bounded complete algebraic poset and the following properties hold:

(1) the mapping $\phi : X \longrightarrow \Sigma\mathcal{A}$, defined by $\phi(x) = N(x), x \in X$ ($N(x)$ is the filter of open neighbourhood of x), is a topological embedding;

(2) $\mathrm{Max}(\mathcal{A}) \subseteq \phi(X)$, and X is T_1 if and only if $\phi(X) = \mathrm{Max}(\mathcal{A})$;

(3) every member of $\mathcal{A}$ is below some $N(x)$, so the closure of $\phi(X)$ in $\Sigma\mathcal{A}$ equals $\mathcal{A}$.

Thus every T_0 space is homeomorphic to a dense subspace of the Scott space of a bounded complete algebraic poset.

A poset P is called a **local** dcpo (or bounded complete dcpo) if every upper bounded directed subset has a supremum [12]. Clearly, every bounded complete poset is a local dcpo.

Lemma 1. *For any local dcpo A, there is a dcpo $\hat{A}$ such that Max(A) and Max$(\hat{A})$ are homeomorphic.*

A poset P is locally quasicontinuous if for each $a \in P$, the sub poset $\downarrow a$ is quasicontinuous.

Lemma 2. *If A is a bounded complete algebraic poset, then the dcpo $\hat{A}$ constructed in Lemma 1 from A is locally quasicontinuous.*

Given a T_1 space, by Theorem 1, there is a bounded complete algebraic poset A such that Max(A) is homeomorphic to X. Since every bounded complete poset is a local dcpo, by Lemma 1 there is a dcpo $\hat{A}$ such that Max(A) is homeomorphic to Max(A). All these deduce the first main result of this paper.

Theorem 2. *Every T_1 topological space has a dcpo model.*

Remark 2. By Lemma 2, we can actually deduce that every T_1 space has a dcpo model that is locally quasicontinuous.

Proposition 1. *Every T_0 space can be embedded, as a dense subset, into to the Scott space of an algebraic dcpo.*

2 Dcpo models of sober spaces

Proposition 2. *If P is a poset such that ΣP is sober, then the subspace Max(P) of ΣP is sober.*

By Proposition III-3.7 of [3], the Scott space of every quasicontinuous dcpo is sober, so we have the following result.

Corollary 1. *For any quasicontinuous dcpo, in particular for any continuous dcpo P, Max(P) is sober.*

Lemma 3. *Let A be a bounded complete algebraic poset and $\hat{A}$ be the dcpo constructed from A in Lemma 1. If Max$(\hat{A})$ is sober then $\Sigma\hat{A}$ is sober.*

From the above two results we deduce the following.

Theorem 3. *A topological space X has a dcpo model whose Scott topology is sober if and only if X is T_1 and sober.*

We call a dcpo P sober, if its Scott topology is sober. Johnstone first constructed a non-sober dcpo in [5], then Isbel gave a non-sober complete lattice [4]. Finding a non-sober dcpo is surprisingly uneasy (as far as the authors know, up-to-date, only three such dcpos have been constructed).

Now if X is a T_1 and non-sober space, then the dcpo model constructed for X in Theorem 2 is non-sober.

For a specific example, let Y be an infinite set and τ be the co-finite topology on Y (i.e. $U \in \tau$ if and only if either $U = \emptyset$ or $Y - U$ is a finite set). Then (Y, τ) is T_1 and non-sober.

Proposition 3. *Let Q be a dcpo model of (Y, τ). Then Q is a non-sober dcpo.*

3 Dcpo models of some special spaces

Let ω_1 be the first non-countable ordinal and $W = [0, \omega_1)$ be the set of all ordinals less than ω_1. Thus W consists of all finite and infinite countable ordinals.

Remark 3. The following facts are well known. 1) $|W| = \aleph_1$.

2) For any countable subset $D \subseteq W$, sup$D \in W$, here the sup$D$ is taken with respect to the usual linear order on ordinals.

3) For any $\alpha \in W$, $\{\beta : \beta \leq \alpha\}$ is a finite or countably infinite subset of W.

Let τ be the co-countable topology on W, that is $U \in \tau$ if and only if either $U = \emptyset$ or $W - U$ is a finite or countably infinite set. We now construct a simpler dcpo model for (W, τ).

Let $P_{\aleph_0} = \{x_\alpha : x \in W, \alpha \in W\} \cup W$. The order on P is defined as follows:
(i) $x_\alpha \leq y_\beta$ iff $\alpha = \beta$ and $x \leq y$;
(ii) $x_\alpha < \alpha$;
(iii) $x_\alpha < \beta$, where $\alpha \neq \beta$, iff $x < \beta$.

Then $P_{\aleph_0}$ is a dcpo and $\mathrm{Max}(P_{\aleph_0}) = W$.

Lemma 4. *(1) For any finite or countably infinite subset $A \subseteq W$, there is a Scott closed set F of $P_{\aleph_0}$ such that $A = F \cap W$.*

(2) For any Scott closed set F of $P_{\aleph_0}$, either $W \subseteq F$ or $W - F$ is at most a countably infinite set.

Proposition 4. *The dcpo $P_{\aleph_0}$ defined above is a model of the space of set $W = [0, \omega_1)$ with the co-countable topology.*

As W is not sober, its dcpo model $P_{\aleph_0}$ is non-sober in the Scott topology. This gives another example of non-sober dcpo.

In general, let $\aleph$ be a cardinal and $W_\aleph$ be the set of all ordinals α with $|\alpha| < \aleph$. The $\aleph$-complementary topology μ on $W_\aleph$ is the topology whose open sets are either $\emptyset$ or whose complement has cardinal less than or equal to $\aleph$. Then we can construct a dcpo model of $(W_\aleph, \tau)$ in a similar way as for (W, τ).

Remark 4. (1) Following the method as for Lemma 4, let $\mathbb{N}$ be the set of all natural numbers and τ the co-finite topology on $\mathbb{N}$. Let $P = \{n_k : n, k \in \mathbb{N}\} \cup \mathbb{N}$. Define the partial order $\leq$ on P by

$$m_k \leq n \text{ for any } k \leq n, n_k \leq m_l \text{ iff } m = n \text{ and } k \leq l.$$

Then P is a dcpo model of $(\mathbb{N}, \tau)$ where τ is the co-finite topology.

(2) In [5], Johnstone gives an example of a dcpo whose Scott topology is not sober (this is the first such example ever constructed). One can verify that this dcpo isomorphic to the dcpo P defined in (1).

A dcpo model P of a T_1 space X is said to satisfy the Lawson condition if X is homeomorphic to $\mathrm{Max}(P)$ with the inherited Lawson topology on P. Lawson proved that a space has a continuous dcpo model satisfying Lawson condition that has a countable base iff the space is Polish[7]. In [11], it was proved that a space has an algebraic poset model satisfying Lawson condition iff it is zero-dimensional.

Theorem 4. *If a space is zero dimensional then it has a dcpo model satisfying Lawson condition.*

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